MAT4450 - Spring 2024 - Exercises - Set 12

Exercise 48

Let $n \in \mathbb{N}$ and consider $\mathcal{A} := \mathcal{B}(\mathcal{H})$ as a C^*-algebra with unit I (= the identity operator).

Assume $A \in \mathcal{A}$ is normal and write $\mathrm{sp}(A) = \{\lambda_1, \ldots, \lambda_k\}$ where $k \leq n$ and $\lambda_j \neq \lambda_{j'}$ for $j \neq j'$.

For each j, let $f_j : \mathrm{sp}(A) \to \mathbb{C}$ denote the indicator function of the set $\{\lambda_j\}$ inside $\mathrm{sp}(A)$, and set $P_j := f_j(A) \in \mathcal{A}$.

We have seen in a lecture that the P_j's are orthogonal projections in $\mathcal{A}$ which are are orthogonal to each other (i.e., $P_j P_j' = 0$ for $j \neq j'$) and satisfy

$$\sum_{j=1}^{k} P_j = I \quad \text{and} \quad \sum_{j=1}^{k} \lambda_j P_j = A.$$

For each j, let $E^A_{\lambda_j}$ denote the eigenspace of A corresponding to the eigenvalue λ_j, i.e., $E^A_{\lambda_j} = \ker\big(\lambda_j I - A\big)$.

Show that each P_j is the orthogonal projection from H onto $E^A_{\lambda_j}$.

Exercise 49

Let H be a complex Hilbert space and consider $\mathcal{A} = \mathcal{B}(H)$ as a C^*-algebra.

Set $\mathcal{A}^+ = \{A \in \mathcal{A} : A \geq 0\}$.

a) Show that the following properties of $\mathcal{A}^+$ hold:

- $\mathcal{A}^+$ is closed in $\mathcal{A}$.
- If $A, B \in \mathcal{A}^+$ and $\lambda \in \mathbb{R}^+$, then $A + B \in \mathcal{A}^+$ and $\lambda A \in \mathcal{A}^+$.
- If $A \in \mathcal{A}^+ \cap (-\mathcal{A}^+)$, then $A = 0$.
- If $A, B \in \mathcal{A}^+$ and $AB = BA$, then $AB \in \mathcal{A}^+$.

b) Give an example with $H = \mathbb{C}^2$ where $A, B \in \mathcal{A}^+$, $AB \neq BA$ and $AB \notin \mathcal{A}^+$.

Exercise 50

Let V be a complex vector space and $L : V \times V \to \mathbb{C}$ be a *sesquilinear form on* V, i.e., L is linear in the first variable and conjugate-linear in the second.

a) Show that

$$L(v, w) = \frac{1}{4} \sum_{k=0}^{3} i^k \, L\big(v + i^k w, v + i^k w\big) \quad \text{for all } v, w \in V.$$

(This is called the *polarization identity*.)

Deduce that if L' is a sesquilinear form on V such that $L'(v, v) = L(v, v)$ for all $v \in V$, then $L' = L$.

b) Define $L^* : V \times V \to \mathbb{C}$ by $L^*(v, w) := \overline{L(w, v)}$ for all $v, w \in V$. Check that L^* is also a sesquilinear form on V.

Deduce that L is *self-adjoint*, that is, $L^* = L$, if and only if $L(v, v) \in \mathbb{R}$ for all $v \in V$.

c) Let H be a complex Hilbert space, $S \in \mathcal{B}(H)$ and let $L_S : H \times H \to \mathbb{C}$ denote the sesquilinear form on H given by $L_S(\xi, \eta) := \langle S(\xi), \eta \rangle$ for all $\xi, \eta \in H$.

Check that $(L_S)^* = L_{S^*}$. Deduce that L_S is self-adjoint if and only if S is self-adjoint, if and only if $\langle S(\xi), \xi \rangle \in \mathbb{R}$ for all $\xi \in H$.