MAT4450 - Spring 2024 - Exercises - Set 13

Exercise 51

Let H be a complex Hilbert space.

a) Assume that $T \in \mathcal{B}(H)$ is normal, $g \in B_b(\mathrm{sp}(T))$ and $f \in C(\Omega)$, where Ω is a compact subset of $\mathbb{C}$ containing $g(\mathrm{sp}(T))$.

Show that $(f \circ g)(T) = f(g(T))$.

b) Let $U \in \mathcal{B}(H)$ be unitary. Use a) to show that there exists a self-adjoint operator $\Theta \in \mathcal{B}(H)$ such that $U = \exp(i\Theta)$.

Exercise 52

Let H be a complex Hilbert space and $T \in \mathcal{B}(H)$ be normal. Let $A \mapsto P(A)$ denote the projection-valued measure associated with T (so $P(A) := 1_A^{\mathrm{sp}(T)}(T)$ for every Borel subset A of $\mathrm{sp}(T)$). Moreover, let $\lambda \in \mathbb{C}$ and set $B(\lambda, \varepsilon) := \{\lambda' \in \mathbb{C} : |\lambda' - \lambda| < \varepsilon\}$.

a) Show that

$$\lambda \in \mathrm{sp}(T) \Leftrightarrow P\big(B(\lambda, \varepsilon) \cap \mathrm{sp}(T)\big) \neq 0 \text{ for all } \varepsilon > 0\,.$$

b) Let $\lambda \in \mathrm{sp}(T)$. Show that

$$\lambda \text{ is an eigenvalue of } T \Leftrightarrow P\big(\{\lambda\}\big) \neq 0,$$

in which case $P\big(\{\lambda\}\big)$ is the orthogonal projection from H onto the eigenspace $E_\lambda^T = \ker(\lambda I - T)$.

Exercise 53

Let H be a complex Hilbert space and $(X, \mathcal{M})$ be a measure space. Assume that $A \mapsto P(A)$ is a map from $\mathcal{M}$ into $\mathcal{B}(H)$ such that $P(A)$ is an orthogonal projection for every $A \in \mathcal{M}$, $P(\emptyset) = 0$, and $P(X) = I$. Consider the following conditions:

i) For every $\xi \in H$ and every sequence $\{A_j\}_{j\in\mathbb{N}}$ of pairwise disjoint sets in $\mathcal{M}$ we have that

$$P\Big(\bigcup_{j=1}^{\infty} A_j\Big)\,\xi \;=\; \sum_{j=1}^{\infty} P(A_j)\,\xi\,.$$

ii) For every $\xi, \eta \in H$ the map $\mu_{\xi,\eta} : \mathcal{M} \to \mathbb{C}$ defined by

$$\mu_{\xi,\eta}(A) = \big\langle P(A)\xi, \eta\big\rangle \quad \text{for all } A \in \mathcal{M}$$

is a complex measure on $(X, \mathcal{M})$.

a) Assume that i) holds. Show that ii) holds, and that $\|\mu_{\xi,\eta}\| \leq \|\xi\|\,\|\eta\|$ for all $\xi, \eta \in H$, where $\|\mu_{\xi,\eta}\| = |\mu_{\xi,\eta}|(\Omega)$ and $|\mu_{\xi,\eta}|$ denotes the total variation of $\mu_{\xi,\eta}$.

b) Assume that i) holds. Show that the following property is satisfied:

$$P(A \cap B) = P(A)P(B) \qquad \text{for all } A, B \in \mathcal{M}.$$

Note that we included this property in our definition of a projection-valued measure on $(X, \mathcal{M})$. It follows from b) that it can be skipped in the definition.