MAT4450 - Spring 2024 - Exercises - Set 14

Exercise 54

Let H be a Hilbert space (over $\mathbb{C}$) and $T \in \mathcal{B}(H)$. Let $T = U\,|T|$ be the polar decomposition of T (so U is a partial isometry such that $\ker(U) = \ker(T)$).

a) Show that the following relations hold:

- $U^*U|T| = |T|$
- $U^*T = |T|$
- $UU^*T = T$
- $|T^*| = U|T|U^*$
- $T^* = U^*|T^*|$, and this gives the polar decomposition of T^*.

b) Assume that T is invertible. Show that U is a unitary.

Exercise 55

Let H be a Hilbert space (over $\mathbb{C}$). Describe the polar decomposition of $T \in \mathcal{B}(H)$ in the following cases:

- $T \geq 0$
- T is an orthogonal projection
- T is a partial isometry
- T is an isometry

Exercise 56

Assume H is a separable infinite-dimensional Hilbert space (over $\mathbb{C}$), and let $\{e_j : j \in \mathbb{N}\}$ be an o.n.b. for H. Let $S \in \mathcal{B}(H)$ denote the (unilateral) shift-operator associated with this basis, so S is determined by $S(e_j) = e_{j+1}$ for all $j \in \mathbb{N}$. Let $n \in \mathbb{N}$. Check that the following facts hold:

- S^n is an isometry with range $H_n := \{e_1, \ldots, e_n\}^\perp$.
- $(S^n)^* = (S^*)^n$ is a partial isometry with initial space H_n and final space H.