MAT4450 - Spring 2024 - Exercises - Set 2

Exercise 6

Let X be a vector space (over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$) and let S be a nonempty family of seminorms on X. Let τ_S denote the weak topology on X induced by S. We recall that τ_S is the weak topology on X determined by the family $\{\rho_{(\sigma,z)}\}_{(\sigma,z)\in S\times X}$ of real-valued functions given by $\rho_{(\sigma,z)}(y) := \sigma(y-z)$ for all $y \in X$.

Let $x \in X$. For $\sigma \in S$ and $\varepsilon > 0$, set $B^{\sigma}_{\varepsilon}(x) := \{y \in X : \sigma(y-x) < \varepsilon\}$.

Moreover, for $z \in X$, set $V_{(\sigma,z),\,\varepsilon}(x) := \{y \in X : |\sigma(y-z) - \sigma(x-z)| < \varepsilon\}$.

a) Check that for any $z \in X$ and $\varepsilon > 0$ we have

$$V_{(\sigma,x),\,\varepsilon}(x) = B^{\sigma}_{\varepsilon}(x) \subseteq\ V_{(\sigma,z),\varepsilon}(x).$$

b) If F is a finite nonempty subset of S and $\varepsilon > 0$, we set

$B^{F}_{\varepsilon}(x) := \bigcap_{\sigma\in F} B^{\sigma}_{\varepsilon}(x) = \{y \in X : \sigma(y-x) < \varepsilon \text{ for all } \sigma \in F\}$.

Use Exercise 5 and part a) to show that the family

$$\mathcal{B}_x := \left\{B^{F}_{\varepsilon}(x) : F \text{ is a finite nonempty subset of } S \text{ and } \varepsilon > 0\right\}$$

is a neighborhood basis at x (for τ_S).

Exercise 7

Let $X = C(\mathbb{R}, \mathbb{C})$ denote the vector space of all continuous complex functions on $\mathbb{R}$, and let $S = \{\sigma_K : K \subseteq \mathbb{R}, K \text{ compact}\}$ be the family of seminorms on X given by

$$\sigma_K(f) := \sup\{|f(t)| : t \in K\} \quad (f \in X)$$

for each $K \subseteq \mathbb{R}, K$ compact.
Show that this family of seminorms is separating, i.e., for each $f \in X, f \neq 0$, there exists some $K \subseteq \mathbb{R}, K$ compact, such that $\sigma_K(f) \neq 0$. (This implies that the topology on $X = C(\mathbb{R}, \mathbb{C})$ induced by this family, i.e., the topology on X of uniform convergence on compact subsets of $\mathbb{R}$, is Hausdorff).

Exercise 8

Let X, X' be vector spaces (over $\mathbb{F}$). Let S (resp. S') denote a family of seminorms on X (resp. X') and let τ (resp. τ') denote the topology on X (resp. X') induced by S (resp. S').

Let $T : X \to X'$ be a linear map, and consider T as map from (X, τ) to (X', τ'). Show that the following statements are equivalent:

a) T is continuous.

b) T is continuous at 0.

c) For each $\sigma' \in S'$ there exist a nonempty finite subset F of S and $M > 0$ such that

$$\sigma'\big(T(x)\big) \ \leq\ M \max_{\sigma\in F}\{\sigma(x)\} \quad \text{for all } x \in X.$$

Note: An immediate consequence is that a linear functional ℓ on X is continuous (w.r.t. τ) if and only if there exist a nonempty finite subset F of S and $M > 0$ such that

$$|\ell(x)| \ \leq\ M \max_{\sigma\in F}\{\sigma(x)\} \quad \text{for all } x \in X.$$

Exercise 9

Let X be topological vector space and let $\ell : X \to \mathbb{F}$ be a linear functional (where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$). Show that the following conditions are equivalent:

a) ℓ is continuous on X.

b) ℓ is continuous at some point of X.

c) $\sup\{\operatorname{Re}\ell(u) \mid u \in U\} < \infty$ for some nonempty open $U \subseteq X$.

d) $\inf\{\operatorname{Re}\ell(u) \mid u \in U\} > -\infty$ for some nonempty open $U \subseteq X$.

e) $\sup\{|\ell(u)| \mid u \in U\} < \infty$ for some nonempty open $U \subseteq X$.

Exercise 10 Let X be an *infinite-dimensional* normed space.

a) Let U be a weakly open neighbourhood of 0 in X (i.e., $0 \in U \subseteq X$ and $U \in \tau_{\text{weak}}$). Show that U contains an infinite-dimensional subspace of X.

b) Show that the weak topology on X does not coincide with the norm topology.