MAT4450 - Spring 2024 - Exercises - Set 3

Exercise 11

Let X be a normed space, $X \neq \{0\}$. We recall that if X is finite-dimensional, with $\dim(X) = n$, then the dual space X^* is finite-dimensional with $\dim(X^*) = n$. We also recall that if B denotes the closed unit ball in X, then B is compact (w.r.t. the norm-topology) if and only if X is finite-dimensional.

a) Assume that X^* is finite-dimensional. Show that X is finite-dimensional too.
(Thus, we get that X is finite-dimensional $\Leftrightarrow$ X^* is finite-dimensional.)

b) Show that X is finite-dimensional if and only if the weak*-topology on X^* agrees with the norm-topology on X^*.

(We recall that the weak*-topology on X^* is the topology τ_{weak^*} determined by the family $\{j_x : x \in X\}$ of linear functionals on X^* defined for each $x \in X$ by $j_x(\varphi) = \varphi(x)$ for all $\varphi \in X^*$.)

c) Show that X is finite-dimensional if and only if the closed unit ball B^* in X^* is compact (w.r.t. the norm-topology on X^*).

d) Assume that X is infinite-dimensional. Let W be a weak*-open neighborhood of 0 in X^* (i.e., $0 \in W \subseteq X^*$ and $W \in \tau_{\text{weak}^*}$). Show that W contains an infinite-dimensional subspace of X^*.

Exercise 12

Let X be a vector space (over $\mathbb{F}$) and let A be a nonempty subset of X. We recall that $a \in A$ is called an internal point of A if for all $x \in X \setminus \{0\}$ there exists some $\varepsilon > 0$ such that $a + \lambda x \in A$ for all $\lambda \in \mathbb{F}$ satisfying $|\lambda| < \varepsilon$. Note that it is not necessary to assume that A is convex for this definition to make sense. We let A^{int} denote the set of all internal points of A.

a) Assume that X is a topological vector space. Show that $A^o \subseteq A^{\text{int}}$, where A^o denotes the interior of A, i.e., the set of all interior points of A.[1]

b) Consider $A = \{(x, y) \in [-1, 1] \times [-1, 1] : x^2 \leq y \text{ or } y \leq 0\} \subseteq \mathbb{R}^2$ and $X = \mathbb{R}^2$ with its usual topology. Check that $A^o \neq A^{\text{int}}$.

c) Assume that X is a finite-dimensional normed space and A is convex. Show that $A^o = A^{\text{int}}$.

d) Consider $X = C([0, 1], \mathbb{R})$ as a normed space (over $\mathbb{R}$) w.r.t. the norm $\|f\|_1 = \int_0^1 |f(t)|\, dt$, and $A = \{f \in X \mid \sup_{t \in [0,1]} |f(t)| < 1\}$. Check that A is convex and $A^o \neq A^{\text{int}}$.

[1] A point $x \in A$ is called an interior point of A if A is a neighbourhood of x.