MAT4450 - Spring 2024 - Exercises - Set 4

Exercise 13

Give an example of two nonempty closed convex subsets F and K of $\mathbb{R}^2$ which don't satisfy the conclusion of the Hahn-Banach separation theorem III (cf. the notes from the lecture on Feb. 13). Note that F and K must both be unbounded.

Exercise 14

Exercise 2.4.6 in Pedersen's book.

Exercise 15

a) Let $n, m \in \mathbb{N}$, and choose some norms on $\mathbb{F}^n$ and $\mathbb{F}^m$. (Since all norms on a finite dimensional vector space are equivalent, it does not matter which norms you choose). Let A be a $m \times n$ matrix over $\mathbb{F}$, and $T : \mathbb{F}^n \to \mathbb{F}^m$ be the linear map having A as its standard matrix. Note that T is automatically bounded.

Identify $\mathbb{F}^n$ with $(\mathbb{F}^n)^*$ via the isomorphism $y \mapsto \ell_y$, where $\ell_y(x) := x \cdot y = \sum_{i=1}^n x_i y_i$ for $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{F}^n$. Similarly, identify $\mathbb{F}^m$ with $(\mathbb{F}^m)^*$.

Check that the standard matrix of the adjoint operator $T^* : \mathbb{F}^m \to \mathbb{F}^n$ is the transpose of A.

b) Let X be a normed space over $\mathbb{F}$, and let $\varphi \in X^*$, i.e., $\varphi \in \mathcal{B}(X, \mathbb{F})$. Identify $\mathbb{F}$ with $(\mathbb{F})^*$ in the obvious way.

Check that the adjoint map $\varphi^* \in \mathcal{B}(\mathbb{F}, X^*)$ is given by $\varphi^*(\lambda) = \lambda\varphi$ for all $\lambda \in \mathbb{F}$.

Exercise 16

Consider $X = \ell^1(\mathbb{N}, \mathbb{F})$ as a normed space w.r.t. the $\| \cdot \|_1$-norm. We recall that $\ell^\infty(\mathbb{N}, \mathbb{F})$, equipped with the $\|.\|_\infty$-norm, may be identified with X^* via the isometric isomorphism $g \mapsto \varphi_g$, where

$$\varphi_g(f) = \sum_{n=1}^{\infty} f(n)g(n)$$

for all $f \in \ell^1(\mathbb{N}, \mathbb{F})$, $g \in \ell^\infty(\mathbb{N}, \mathbb{F})$.

Set $Y := c_0(\mathbb{N}, \mathbb{F}) = \{g \in \ell^\infty(\mathbb{N}, \mathbb{F}) \mid \lim_{n\to\infty} g(n) = 0\}$, and equip Y with the $\|.\|_\infty$-norm. We also recall that X may be identified with Y^*, via the isometric isomorphim $f \mapsto \psi_f$, where

$$\psi_f(g) = \sum_{n=1}^{\infty} f(n)g(n)$$

for all $f \in X = \ell^1(\mathbb{N}, \mathbb{F})$, $g \in Y = c_0(\mathbb{N}, \mathbb{F})$.

a) Define a linear map $T : X \to Y$ by

$$[T(f)](n) = \sum_{m=n}^{\infty} f(m)$$

for all $f \in X$ and $n \in \mathbb{N}$. Check that T is bounded. Then find an expression for $T^* \in \mathcal{B}(Y^*, X^*) = \mathcal{B}(X, X^*)$.

b) Consider Y as a subspace of X^*. Check that Y is norm-closed in X^*. Then show that $(Y^\perp)^\perp = X^*$, and conclude that $Y \neq (Y^\perp)^\perp$.