MAT4450 - Spring 2020 - Exercises - Set 5

Exercise 17

Consider $\ell^1 := \ell^1(\mathbb{N}, \mathbb{F})$ as a Banach space with the $\|\cdot\|_1$-norm, and define $T : \ell^1 \to \ell^1$ by

$$[T(f)](n) = \frac{1}{n} f(n), \quad n \in \mathbb{N}$$

for all $f \in \ell^1$.

a) Check that T is well-defined, injective and bounded, with $\|T\| = 1$.

b) Show that the range of T, i.e., $Z := T(\ell^1)$, is not closed in ℓ^1.

c) Since T is a bijection between ℓ^1 and Z, we may consider the inverse map $T^{-1} : Z \to \ell^1$. Show that T^{-1} is unbounded. Why does this imply that Z is not closed ?

Exercise 18

Solve Exercise 2.3.8 in Pedersen's book.

Exercise 19

Solve Exercise 2.4.4 in Pedersen's book.

Exercise 20 [NB: This exercise will be part of the compulsory assignment]

Let M be a closed subspace of a locally convex Hausdorff topological vector space (X, τ) and let $\psi : M \to \mathbb{F}$ be a continuous linear functional. Show that there exists some $\varphi \in (X, \tau)^*$ which extends ψ, that is, we have $\varphi_{|M} = \psi$.

Note that this result is another version of the Hahn-Banach extension theorem.

Hint: Consider $N = \{x \in M : \psi(x) = 0\}$. If $\psi \neq 0$, then one may find $x_0 \in M$ such that $\psi(x_0) = 1$, so $x_0 \in M \setminus N \subseteq X \setminus N$. Use one version of the Hahn-Banach separation theorem.

Exercise 21

Let H be a Hilbert space and M be a closed subspace of H. Let $M^\perp$ denote the orthogonal complement of M in H.[1]

Consider the quotient space H/M with the quotient norm. Show that H/M is isometrically isomorphic with $M^\perp$. Then explain how H/M may be organized as a Hilbert space in a natural way.

Exercise 22

Let X be a normed space. Let M be a closed subspace of X and let $Q : X \to X/M$ denote the quotient map, i.e., $Q(x) = x + M$ for all $x \in X$. We have seen that Q is bounded. Show that the adjoint operator $Q^* : (X/M)^* \to X^*$ is isometric.

[1]Note that if one identifies H with H^* via the conjugate linear isometric map $y \mapsto \varphi_y$, where $\varphi_y(x) = \langle x, y \rangle$ for all $x \in H$, then $M^\perp$ coincides with the anihilator of M.