MAT4450 - Spring 2024 - Exercises - Set 6

Exercise 23

Let X be a vector space (over $\mathbb{F}$).

a) Let A be a nonempty subset of X and set

$$\mathrm{co}(A) = \Big\{ \sum_{j=1}^{n} \lambda_j \, a_j \mid n \in \mathbb{N},\, \lambda_1, \ldots, \lambda_n \in [0,1],\, \sum_{j=1}^{n} \lambda_j = 1,\, a_1, \ldots, a_n \in A \Big\}.$$

Show that $\mathrm{co}(A)$, the convex hull of A, is the least convex subset of X containing A.

b) Let C be a nonempty convex subset of X. Assume that F is a face of C and K is a face of F. Show that K is a face of C.

Exercise 24

Let Ω be a compact Hausdorff space and let $\{f_n\}$ be a sequence in $C(\Omega, \mathbb{F})$ such that $\sup_{n \in \mathbb{N}} \|f_n\|_\infty < \infty$ and $\{f_n\}$ converges to some $f \in C(\Omega, \mathbb{F})$ pointwise on Ω. Show that

$$f \in \overline{\mathrm{co}\big(\{f_n, n \in \mathbb{N}\}\big)}^{\|\cdot\|_\infty}.$$

Exercise 25 [NB: this exercise will be part of the compulsory assignment]

We recall that if C is a convex subset of a vector space, then $\mathrm{ex}(C)$ denotes the set of all extreme points of C.

Let H be a complex Hilbert space $\neq \{0\}$. Set

$$B := \{\xi \in H \,:\, \|\xi\| \leq 1\} \text{ and } \mathcal{B} := \{T \in \mathcal{B}(H) \,:\, \|T\| \leq 1\}.$$

a) Show that $\mathrm{ex}(B) = \{\eta \in B \,:\, \|\eta\| = 1\}$.

b) Let $T \in \mathcal{B}(H)$. Assume that T or T^* is isometric.[1] Show that $T \in \mathrm{ex}(\mathcal{B})$.

Exercise 26

Consider the Banach space $X = (L^1(\mathbb{R}, \mathcal{B}_\mathbb{R}, \mu), \|\cdot\|_1)$ (over $\mathbb{F}$), where $\mathcal{B}_\mathbb{R}$ denotes the σ-algebra of all Borel subsets of $\mathbb{R}$ and μ denotes the Lebesgue measure on $\mathcal{B}_\mathbb{R}$.

Consider $B := \{f \in X : \|f\|_1 \leq 1\}$, which is clearly convex.

a) Show that the convex ball B has no extreme points.

Hint. Assume for contradiction that there exists some $f \in B$. Show first that $\|f\|_1 = 1$. Then consider the continuous function $F(t) := \int_{(-\infty, t]} |f| \, d\mu, \; t \in \mathbb{R}$.

b) Deduce that there is no topology on X making it a locally convex Hausdorff topological vector space such that B is compact. Explain why this implies that $(L^1(\mathbb{R}, \mathcal{B}_\mathbb{R}, \mu), \|\cdot\|_1)$ can not be isomorphic to the dual space of $(L^\infty(\mathbb{R}, \mathcal{B}_\mathbb{R}, \mu), \|\cdot\|_\infty)$.

Exercise 27

Let (X, τ) be a locally convex Hausdorff topological vector space and K be a nonempty compact convex subset of X.
Let $\varphi \in (X, \tau)^*$ and set $m := \inf \mathrm{Re}\, \varphi(K)$, $M := \sup \mathrm{Re}\, \varphi(K)$, $s := \sup |\varphi(K)|$.

Show that there exist $x, y, z \in \mathrm{ex}(K)$ such that

$$\mathrm{Re}\, \varphi(x) = m, \; \mathrm{Re}\, \varphi(y) = M, \; |\varphi(z)| = s\,.$$

[1] By T^* we mean here the (Hilbert) adjoint operator of T as defined for a bounded operator on a Hilbert space.