MAT4450 - Spring 2024 - Exercises - Set 7

Exercise 28

Let Ω be a nonempty compact subset of $\mathbb{R}^2$ and consider the set $\mathcal{P}$ of all real polynomials in two commuting variables as a subset of $C(\Omega, \mathbb{R})$. Show that $\mathcal{P}$ is dense in $C(\Omega, \mathbb{R})$ (w.r.t. $\|\cdot\|_\infty$).

Exercise 29

Let Ω be a compact Hausdorff space, and let $\mathcal{A}$ be a subalgebra of $C(\Omega, \mathbb{R})$.

a) Show that $|f|^{1/n} \in \overline{\mathcal{A}}^{\|\cdot\|_\infty}$ for every $f \in \mathcal{A}$ and $n \in \mathbb{N}$.

b) Assume that $\mathcal{A}$ separates the points of Ω and that there exists some $g \in \mathcal{A}$ such that $g(x) \neq 0$ for all $x \in \Omega$. Show that $\overline{\mathcal{A}}^{\|\cdot\|_\infty} = C(\Omega, \mathbb{R})$.

Exercise 30.

Let H denote a nontrivial (complex) Hilbert space. Recall that $\mathcal{K}(H)$ denotes the set of all compact linear operators on H, and that it is a Banach algebra (w.r.t. operator norm).

Show that $\mathcal{K}(H)$ is unital if and only if H is finite-dimensional.

Exercise 31.

a) Consider the set

$$C_0(\mathbb{R}, \mathbb{C}) := \{f : \mathbb{R} \to \mathbb{C} \mid f \text{ is continuous and } \lim_{t\to\infty} f(t) = \lim_{t\to-\infty} f(t) = 0\}$$

Show that $C_0(\mathbb{R}, \mathbb{C})$ is a norm-closed subalgebra of $C_b(\mathbb{R}, \mathbb{C})$,[1] hence that $C_0(\mathbb{R}, \mathbb{C})$ is a commutative Banach algebra (w.r.t. $\|\cdot\|_\infty$). Check also that $C_0(\mathbb{R})$ is non-unital.

b) More generally, let Ω denote a locally compact Hausdorff space (e.g. $\Omega = \mathbb{R}^n$ with its standard topology). We say that a function $f : \Omega \to \mathbb{C}$ *vanishes at infinity* if for every $\varepsilon > 0$ the set $\{x \in \Omega : |f(x)| \geq \varepsilon\}$ is compact. Set

$$C_0(\Omega, \mathbb{C}) := \{f : \Omega \to \mathbb{C} \mid f \text{ is continuous and vanishes at infinity}\}.$$

Show that $C_0(\Omega, \mathbb{C})$ is a norm-closed subalgebra of $C_b(\Omega, \mathbb{C})$, hence that $C_0(\Omega, \mathbb{C})$ is a commutative Banach algebra (w.r.t. $\|\cdot\|_\infty$). Then show that $C_0(\Omega, \mathbb{C})$ is unital if and only if Ω is compact.

[1] We recall that if Ω is a topological space, then $C_b(\Omega, \mathbb{C}) = \{f : \Omega \to \mathbb{C} : f \text{ is continuous and bounded}\}$ is a commutative Banach algebra (w.r.t. $\|\cdot\|_\infty$).