MAT4450 - Spring 2020 - Exercises - Set 8

Exercise 32

Consider the commutative Banach algebra $\mathcal{A} = C(\Omega, \mathbb{F})$, where Ω is a compact Hausdorff space. Let $\omega_0 \in \Omega$ and define $\varphi : \mathcal{A} \to \mathbb{F}$ by $\varphi(f) = f(\omega_0)$ for $f \in \mathcal{A}$.

a) Check that φ is a continuous algebra-homomorphism from $\mathcal{A}$ into $\mathbb{F}$ (considered as a Banach algebra) satisfying $\|\varphi\| = 1$.

b) Consider the closed ideal of $\mathcal{A}$ given by $\mathcal{J} = \ker \varphi$. Show that the Banach algebra $\mathcal{A}/\mathcal{J}$ is isometrically isomorphic to $\mathbb{F}$.

Note: Two algebras are said to be *isomorphic* if there exists a bijective algebra-homomorphism between them; the inverse map is then necessarily an algebra-homomorphism, as one can easily check. Two Banach algebras are said to be *isomorphic* if there exists a bijective continuous algebra-homomorphism between them; the inverse map is then an algebra-homomorphism which is continuous (as follows from the open mapping theorem). Note that such an isomorphism between Banach algebras is not isometric in general.

Exercise 33

Let $\mathcal{A}$ be a non-unital Banach algebra over $\mathbb{F}$. Set $\widetilde{\mathcal{A}} = \big\{(a, \alpha) \mid a \in \mathcal{A}, \alpha \in \mathbb{F}\big\}$ and define addition, multiplication by scalars and product by

$$\begin{aligned}(a, \alpha) + (b, \beta) &= (a + b, \alpha + \beta),\\ \lambda\,(a, \alpha) &= (\lambda a, \lambda\alpha),\\ (a, \alpha)(a, \beta) &= (ab + \alpha b + \beta a, \alpha\beta)\end{aligned}$$

for $(a, \alpha), (b, \beta) \in \widetilde{\mathcal{A}}$ and $\lambda \in \mathbb{F}$. Check for yourself that $\widetilde{\mathcal{A}}$ becomes a unital algebra with unit $\tilde{e} := (0, 1)$, and that the map $(a, \alpha) \mapsto \alpha$ is an algebra-homomorphism from $\widetilde{\mathcal{A}}$ into $\mathbb{F}$, with kernel equal to $\mathcal{A}_0 := \{(a, 0) \mid a \in \mathcal{A}\}$.

It is usual to identify $\mathcal{A}$ with the ideal $\mathcal{A}_0$ and to write $a + \alpha\,\tilde{e}$ instead of (a, α).

We define a norm on $\widetilde{\mathcal{A}}$ by

$$\|a + \alpha\tilde{e}\| := \|a\| + |\alpha|$$

for every $a + \alpha\tilde{e} \in \widetilde{\mathcal{A}}$.

Verify that $\widetilde{\mathcal{A}}$ becomes a unital Banach algebra w.r.t. $\|\cdot\|$, containing $\mathcal{A}$ as a closed ideal. Check also that the quotient Banach algebra $\widetilde{\mathcal{A}}/\mathcal{A}$ is isometrically isomorphic to $\mathbb{F}$.

Note: From now on, we will use the convention that a unital Banach algebra $\mathcal{A}$ always mean a Banach algebra over $\mathbb{C}$ having a unit $1_{\mathcal{A}}$ such that $\|1_{\mathcal{A}}\| = 1$.

Exercise 34

Let $\mathcal{A}$ be a unital Banach algebra. Recall that $\mathrm{GL}(\mathcal{A}) = \{\ a \in \mathcal{A} : a$ is invertible in $\mathcal{A}\}$.

a) Let $a \in \mathrm{GL}(\mathcal{A})$ and note that $0 \notin \mathrm{sp}_{\mathcal{A}}(a)$. Show that

$$\mathrm{sp}_{\mathcal{A}}(a^{-1}) = \Big\{\ \lambda^{-1} \mid \lambda \in \mathrm{sp}_{\mathcal{A}}(a)\ \Big\}.$$

b) Let $\mathcal{B}$ be unital Banach algebra and assume that $\phi : \mathcal{A} \to \mathcal{B}$ is an algebra-isomorphism such that $\phi(1_{\mathcal{A}}) = \phi(1_{\mathcal{B}})$. Show that $\mathrm{sp}_{\mathcal{A}}(a) = \mathrm{sp}_{\mathcal{B}}(\phi(a))$ for all $a \in \mathcal{A}$.

Exercise 35

Consider the complex Hilbert space $H = L^2([0,1], \mathcal{B}_{[0,1]}), \mu)$, where μ denotes the Lebesgue measure on the Borel σ-algebraen $\mathcal{B}_{[0,1]}$.

Set $\mathcal{A} := \mathcal{B}(H)$, and let $M \in \mathcal{A}$ denote the multiplication operator given by

$$M(g)](t) = tg(t) \qquad \text{for all } g \in H \text{ and } t \in [0,1].$$

Show that $\mathrm{sp}_{\mathcal{A}}(M) = [0,1]$ and that M has no eigenvalues.

Exercise 36

Let S be a nonempty set and consider the unital Banach algebra $\mathcal{A} = \ell^\infty(S, \mathbb{C})$.

Let $f \in \mathcal{A}$. Show that $\mathrm{sp}(f) = \overline{f(S)}$.