MAT4450 - Spring 2024 - Exercises - Set 9

Exercise 37

Solve Exercise 4.1.3 in Pedersen's book.

Exercise 38

Let X be a complex Banach space and consider the Banach algebra $\mathcal{A} := \mathcal{B}(X)$. Let $T \in \mathcal{A}$. The adjoint operator T^* belongs then to the Banach algebra $\mathcal{B} := \mathcal{B}(X^*)$.

Show that $\mathrm{sp}_{\mathcal{B}}(T^*) = \mathrm{sp}_{\mathcal{A}}(T)$.

Exercise 39

Let H denote a nontrivial complex Hilbert space and consider $\mathcal{A} := \mathcal{B}(H)$ as a Banach algebra. Let $\mathcal{B} = \{e_j\}_{j\in J}$ be an orthonormal basis for H and $f \in \ell^\infty(J)$. Set $\lambda_j := f(j) \in \mathbb{C}$ for each $j \in J$. Let $D \in \mathcal{A}$ denote the associated "diagonal" operator satisfying that $D(e_j) = \lambda_j\, e_j$ for every $j \in J$. We have seen in a lecture that

$$\mathrm{sp}_{\mathcal{A}}(D) = \overline{\{\lambda_j \mid j \in J\}} = \overline{f(J)}.$$

Show that $r_{\mathcal{A}}(D) = \|D\| = \|f\|_\infty$.

Exercise 40

Consider the Banach algebra $\mathcal{A} = M_2(\mathbb{C}) \simeq \mathcal{B}(\mathbb{C}^2)$. Give an example of a matrix $A \in \mathcal{A}$ satisfying that $r_{\mathcal{A}}(A) < \|A\|$.

Exercise 41 [NB: This exercise is a part of the compulsory assignment]

Solve Exercise 4.1.6 in Pedersen's book.

Exercise 42

Let $\mathcal{A}$ denote a complex unital Banach algebra with unit $1_{\mathcal{A}}$ satisfying $\|1_{\mathcal{A}}\| = 1$. Let $a \in \mathcal{A}$ and let f be a complex polynomial given by $f(z) = \sum_{k=0}^n c_k z^k$ for some $c_0, c_1, \ldots, c_n \in \mathbb{C}$.

We can then define $f(a) \in \mathcal{A}$ by $f(a) := \sum_{k=0}^n c_k a^k$. It follows from a lemma proved this week that

$$f(\mathrm{sp}_{\mathcal{A}}(a)) \subseteq \mathrm{sp}_{\mathcal{A}}\big(f(a)\big).$$

Show that the reverse inclusion holds, hence that we have $f(\mathrm{sp}_{\mathcal{A}}(a)) = \mathrm{sp}_{\mathcal{A}}\big(f(a)\big)$.