MAT4450 - Spring 2024 - Solutions of exercises - Set 1

Exercise 1. *Let X be a topological space which is* first countable, *that is, every $x \in X$ has a countable neighborhood basis. Let $A \subseteq X$ and $x \in X$. Then we have that*

$$x \in \overline{A} \Leftrightarrow \text{there exists a sequence } \{x_n\} \text{ in } A \text{ such that } x_n \to x \text{ as } n \to \infty.$$

We know that in any topological space the implication ($\Leftarrow$) holds for a net, hence for a sequence. So it suffices to show that ($\Rightarrow$) holds. Assume that $x \in \overline{A}$. Let $\{B_n : n \in \mathbb{N}\}$ denote a neighborhood basis at x. By replacing each B_n by $\cap_{j=1}^n B_j$ if necessary, we may assume that $B_{n+1} \subseteq B_n$ for each $n \in \mathbb{N}$. Now, for each $n \in \mathbb{N}$, we have that $B_n \cap A \neq \emptyset$, so we may pick $x_n \in B_n \cap A$. Consider now any $V \in \mathcal{N}_x$. Choosing n_0 such that $B_{n_0} \subseteq V$, we get that $x_n \in B_n \subseteq B_{n_0} \subseteq V$, hence $x_n \in V$, for every $n \geq n_0$. Thus, $x_n \to x$ as $n \to \infty$, as desired.

Exercise 2. *Let X consists of all real-valued functions on $\mathbb{R}$ and equip it with the topology of pointwise convergence. We recall that a neighborhood basis $\mathcal{B}_f$ at $f \in X$ is given by*

$$\mathcal{B}_f = \{V_{f,\,S,\,\varepsilon} : S \text{ is a finite subset of } \mathbb{R},\, \varepsilon > 0\},$$

where $V_{f,\,S,\,\varepsilon} := \{g \in X : |f(s) - g(s)| < \varepsilon$ for every $s \in S\}$. For $h \in X$ we set $\mathrm{supp}(h) := \{t \in \mathbb{R} : h(t) \neq 0\}$. *Moreover, we set $A := \{h \in X : \mathrm{supp}(h)$ is finite$\}$.*

a) We show that $\overline{A} = X$ in two different ways:

i) Let $f \in X$. Consider $V \in \mathcal{N}_f$. We may then find some $V_{f,\,S,\,\varepsilon} \in \mathcal{B}_f$ such that $V_{f,\,S,\,\varepsilon} \subseteq V$. Define $f_S \in A$ by $f_S(t) = f(t)$ when $t \in S$, and $f_S(t) = 0$ when $t \notin S$. Then we clearly get that $f_S \in (V_{f,S,\,\varepsilon} \cap A) \subseteq (V \cap A)$, so $V \cap A \neq \emptyset$. Thus, $f \in \overline{A}$.

ii) Let $f \in X$. For every finite subset S of X, i.e., $S \in \mathcal{P}_{\mathrm{fin}}(X)$, define f_S as above. Considering $\Lambda := \mathcal{P}_{\mathrm{fin}}(X)$ as a directed set w.r.t. inclusion, we get a net $\{f_S\}_{S \in \Lambda}$ in A, which converges to f:

Indeed, let $V \in \mathcal{N}_f$. Choose $V_{f,\,S,\,\varepsilon} \in \mathcal{B}_f$ such that $V_{f,\,S,\,\varepsilon} \subseteq V$. Then for every $S' \in \Lambda$ such that $S \subseteq S'$ we have that $f_{S'} \in V_{f,\,S,\,\varepsilon} \subseteq V$ (since $|f(s) - f_{S'}(s)| = |f(s) - f(s)| = 0 < \varepsilon$ for all $s \in S$). Thus, again, $f \in \overline{A}$.

b) Let f be any function in $X = \overline{A}$ such that $\mathrm{supp}(f)$ *is uncountable. Then there is no sequence in A which converges to f.*

Assume (for contradiction) that there exists a sequence $\{f_n\}$ in A such that $f_n \to f$ as $n \to \infty$. Set $T := \cup_{n \in \mathbb{N}} \mathrm{supp}(f_n)$. Then T is countable. Moreover, $\mathrm{supp}(f) \subseteq T$. Indeed, let $x \in \mathrm{supp}(f)$. We may then find $n \in \mathbb{N}$ such that $|f(x) - f_n(x)| < |f(x)|/2$. This implies that $|f_n(x)| > |f(x)|/2 > 0$. Thus, $x \in \mathrm{supp}(f_n) \subseteq T$. Since T is countable, we get that $\mathrm{supp}(f)$ is countable, contradicting the hypothesis.

Exercise 3. *Let X be a nonempty set and $A \subseteq X$. Consider a net $\{x_\alpha\}_{\alpha \in \Lambda}$ in X. Then the following statements are true.*

a) If $\{x_\alpha\}$ is eventually in A, then it is frequently in A:

Assume $\{x_\alpha\}$ is eventually in A, i.e., there exists $\alpha_0 \in \Lambda$ such that $x_\alpha \in A$ for all $\alpha \geq \alpha_0$. Let $\beta \in \Lambda$. We may then find $\gamma \in \Lambda$ such that $\gamma \geq \alpha_0$ and $\gamma \geq \beta$. This gives that $x_\gamma \in A$. Hence, $\{x_\alpha\}$ is frequently in A.

b) $\{x_\alpha\}$ is not frequently in A if and only if it is eventually in $X \setminus A$:

We have that $\{x_\alpha\}$ is not frequently in A if and only if there exists some $\alpha_0 \in \Lambda$ such that $x_\beta \notin A$ for all $\beta \geq \alpha_0$, if and only if $\{x_\alpha\}$ is eventually in $X \setminus A$.

c) If $\{x_\alpha\}$ is eventually in A, then it is not eventually in $X \setminus A$, and if $\{x_\alpha\}$ is eventually in $X \setminus A$, then it is not eventually in A:

If $\{x_\alpha\}$ is eventually in $X \setminus A$, then it follows from b) that $\{x_\alpha\}$ is not frequently in A, and then a) gives that $\{x_\alpha\}$ is not eventually in A. This shows that the second assertion holds for every subset A of X. Applying this assertion to the subset $X \setminus A$, we get that the first assertion holds too.

Exercise 4. *Let X be a topological space and let $\{x_\alpha\}_{\alpha\in\Lambda}$ be a universal net in X (so it is eventually in A or eventually in $X \setminus A$ for every $A \subseteq X$). Assume that $x \in X$ is a cluster point of $\{x_\alpha\}$, that is, $\{x_\alpha\}$ is frequently in V for every $V \in \mathcal{N}_x$.*

Then $\{x_\alpha\}$ converges to x:

Consider $V \in \mathcal{N}_x$. Then $\{x_\alpha\}$ is frequently in V, so, using b) in the previous exercise, we get that $\{x_\alpha\}$ is not eventually in $X \setminus V$. By universality of the net $\{x_\alpha\}$, we get that it is eventually in V. Thus, $x_\alpha \to x$.

About **Exercise 1.3.3** in Pedersen's book.

For each λ let us denote by x_λ the sum defined on the left (using the inf) and by y_λ the sum defined on the right (using the sup). Then we clearly have that

$$m(b-a) \le x_\lambda \le y_\lambda \le M(b-a),$$

where $m := \inf\{f(x) : a \le x \le b\}$, $M := \sup\{f(x) : a \le x \le b\}$. Moreover, one checks that the net $\{x_\lambda\}$ is non-decreasing, while $\{y_\lambda\}$ is non-increasing. It follows that both these nets are convergent, with

$$m(b-a) \le \lim_\lambda x_\lambda \le \lim_\lambda y_\lambda \le M(b-a).$$

Now, if one says that the function f is R-integrable when $\lim_\lambda x_\lambda = \lim_\lambda y_\lambda$, then one has to work a bit to show that this is equivalent to f being Riemann-integrable. As this is somewhat outside the scope of this course, we skip the details (some of which can be found in Enstad's notes).

Exercise 5

Let X be a nonempty set and let $\{\rho_i\}_{i\in I}$ be a nonempty family of functions from X to $\mathbb{R}$. Consider $\mathbb{R}$ as a topological space w.r.t. to its standard topology and let τ denote the weak topology on X determined by $\{\rho_i\}_{i\in I}$, i.e., τ is the topology on X generated by the family $\mathcal{E} = \{\rho_i^{-1}(U) : i \in I, U$ is an open subset of $\mathbb{R}\} \subseteq \mathcal{P}(X)$.

Let $x \in X$. For $i \in I$ and $\varepsilon > 0$, set $V_{i,\varepsilon}(x) = \big\{y \in X : |\rho_i(y) - \rho_i(x)| < \varepsilon\big\} \subseteq X$. Moreover, set $\mathcal{U}_x = \Big\{\bigcap_{k=1}^n V_{i_k,\varepsilon_k}(x) : n \in \mathbb{N}, i_1, \dots, i_n \in I \text{ and } \varepsilon_1, \dots, \varepsilon_n > 0\Big\} \subseteq \mathcal{P}(X)$.

Then $\mathcal{U}_x$ is a neighborhood basis at x (for τ):

For $i \in I, a \in \mathbb{R}, \varepsilon > 0$, set

$$B_{i,a,\varepsilon} := \rho_i^{-1}((a-\varepsilon, a+\varepsilon)) = \{y \in X : |\rho_i(y) - a| < \varepsilon\}.$$

Since $\{(a-\varepsilon, a+\varepsilon) : a \in \mathbb{R}, \varepsilon > 0\}$ is a basis for the standard topology on $\mathbb{R}$, it follows readily that τ is generated by the family $\mathcal{E}' = \{B_{i,a,\varepsilon} : i \in I, a \in \mathbb{R}, \varepsilon > 0\}$. A basis $\mathcal{B}$ for τ is then given by the family of all finite intersections of sets in $\mathcal{E}'$, and a neighborhood basis $\mathcal{B}_x$ at x for τ is therefore given by the family $\mathcal{B}_x = \{B \in \mathcal{B} : x \in B\}$. This means that $W \in \mathcal{N}_x$ if and only if there exists some $B \in \mathcal{B}_x$ such that $B \subseteq W$. To show that $\mathcal{U}_x$ is a neighborhood basis at x, i.e., that $W \in \mathcal{N}_x$ if and only if there exists some $U \in \mathcal{U}_x$ such that $U \subseteq W$, it therefore suffices to show that the following two assertions hold:

1) For every $U \in \mathcal{U}_x$, there exists $B \in \mathcal{B}_x$ such that $B \subseteq U$.

2) For every $B \in \mathcal{B}_x$, there exists $U \in \mathcal{U}_x$ such that $U \subseteq B$.

Since $V_{i,\varepsilon}(x) = B_{i,\,\rho_i(x),\,\varepsilon}$, we have that $\mathcal{U}_x \subseteq \mathcal{B}_x$, so it is obvious that 1) holds.

On the other hand, let $B \in \mathcal{B}_x$, so $B = \bigcap_{k=1}^{n} B_{i_k,\,a_k,\,\varepsilon_k}$ for some $n \in \mathbb{N}, i_1, \ldots, i_n \in I$, $a_1, \ldots, a_n \in \mathbb{R}$ and $\varepsilon_1, \ldots, \varepsilon_n > 0$. Since $x \in B$ we have $|\rho_{i_k}(x) - a_k| < \varepsilon_k$ for each $k = 1, \ldots, n$, so $\delta_k := \varepsilon_k - |\rho_{i_k}(x) - a_k| > 0$ for each $k = 1, \ldots, n$. Set now $U := \cap_{k=1}^{n} V_{i_k,\,\delta_k}(x) \in \mathcal{U}_x$. Then we have $U \subseteq B$. Indeed, if $y \in U$, then we have

$$|\rho_{i_k}(y) - a_k| \leq |\rho_{i_k}(y) - \rho_{i_k}(x)| + |\rho_{i_k}(x) - a_k| < \delta_k + |\rho_{i_k}(x) - a_k| = \varepsilon_k$$

for every $k = 1, \ldots, n$, showing that $y \in B$, as desired. This shows that 2) holds.