MAT4450 - Spring 2024 - Solutions of exercises - Set 10

Exercise 43 [= Exercise 4.1.13 in Pedersen's book]

Let $\mathcal{A}$ be a unital Banach algebra. Recall that for each $a \in \mathcal{A}$, $\exp(a) \in \mathcal{A}$ is given by

$$\exp(a) = \sum_{n=0}^{\infty} \frac{1}{n!}\, a^n$$

as this power series in a is absolutely convergent in $\mathcal{A}$.

Assume that $a, b \in \mathcal{A}$ commute, i.e., $ab = ba$. Then we have that

$$\exp(a+b) = \exp(a)\exp(b).$$

It is not difficult to give an informal proof of this fact if one is willing to overlook the problem of interchanging the order of summation in a double infinite sum of elements in $\mathcal{A}$. Indeed, on one side, using the binomial formula, we get

$$\begin{aligned}\exp(a+b) &= \sum_{n=0}^{\infty} \frac{1}{n!}\,(a+b)^n \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{1}{k!(n-k)!}\, a^k b^{n-k}\end{aligned}$$

On the other side, we have

$$\begin{aligned}\exp(a)\exp(b) &= \Big(\sum_{k=0}^{\infty} \frac{1}{k!}\, a^k\Big)\Big(\sum_{m=0}^{\infty} \frac{1}{m!}\, b^m\Big) \\ &= \sum_{k=0}^{\infty} \Big(\frac{1}{k!}\, a^k\, \big(\sum_{m=0}^{\infty} \frac{1}{m!}\, b^m\big)\Big) \\ &= \sum_{k=0}^{\infty} \big(\frac{1}{k!}\, a^k\, \big(\sum_{n=k}^{\infty} \frac{1}{(n-k)!}\, b^{n-k}\big)\Big) \\ &= \sum_{k=0}^{\infty}\sum_{n=k}^{\infty} \frac{1}{k!(n-k)!}\, a^k b^{n-k}.\end{aligned}$$

If we are allowed to change the order of summation in these sums, we see that both sides are equal. In order to formalize this, let $\varphi \in \mathcal{A}^*$. By continuity, we get that

$$\varphi\big(\exp(a+b)\big) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{1}{k!(n-k)!}\, \varphi(a^k b^{n-k}),$$

while

$$\varphi\big(\exp(a)\exp(b)\big) = \sum_{k=0}^{\infty}\sum_{n=k}^{\infty} \frac{1}{k!(n-k)!}\, \varphi(a^k b^{n-k}).$$

Set $D := \big\{(n,k) : n \in \mathbb{N} \cup \{0\}, k \in \{0, \ldots, n\}\big\} = \big\{(n,k) : k \in \mathbb{N} \cup \{0\}, n \in \{k, k+1, \ldots\}\big\}$, and define $f : D \to \mathbb{C}$ by

$$f(n,k) := \frac{1}{k!(n-k)!}\, \varphi(a^k b^{n-k}).$$

Then, using Tonelli's theorem, we get that

$$\begin{aligned}\sum_{(n,k)\in D} |f(n,k)| &\leq \|\varphi\| \sum_{(n,k)\in D} \frac{1}{k!(n-k)!} \|a\|^k \|b\|^{n-k} \\ &= \|\varphi\| \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{1}{k!(n-k)!} \|a\|^k \|b\|^{n-k} \\ &= \|\varphi\| \sum_{n=0}^{\infty} \frac{1}{n!} (\|a\| + \|b\|)^n \\ &= \|\varphi\| \exp\big(\|a\| + \|b\|\big) < \infty.\end{aligned}$$

Thus, we may now use Fubini's theorem and obtain that

$$\sum_{(n,k)\in D} f(n,k) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} f(n,k) = \sum_{k=0}^{\infty}\sum_{n=k}^{\infty} f(n,k).$$

This shows that $\varphi\big(\exp(a+b)\big) = \varphi\big(\exp(a)\exp(b)\big)$. Since this holds for every $\varphi \in \mathcal{A}^*$, and $\mathcal{A}^*$ separates $\mathcal{A}$, we get that $\exp(a+b) = \exp(a)\exp(b)$, as desired.

It follows that $\exp(a) \in \mathrm{GL}(\mathcal{A})$ *for every* $a \in \mathcal{A}$:

Indeed, since $-a$ commutes with a, we get that

$$\exp(a)\exp(-a) = \exp(a-a) = \exp(0) = I = \exp(-a+a) = \exp(-a)\exp(a),$$

hence that $\exp(a)$ is invertible in $\mathcal{A}$, with $\exp(a)^{-1} = \exp(-a)$, for every $a \in \mathcal{A}$.

Exercise 41

Consider the unital commutative Banach algebra $\mathcal{A} = \ell^1(\mathbb{Z}, \mathbb{C})$ *(w.r.t. the* $\|\cdot\|_1$*-norm).*
Set $\mathcal{B} := \{f \in \mathcal{A} : f(n) = 0$ *for all* $n < 0\}$.

a) *We have that* $\mathcal{B}$ *is a norm-closed subalgebra of* $\mathcal{A}$ *which contains the unit of* $\mathcal{A}$ *(hence* $\mathcal{B}$ *is also a unital commutative Banach algebra)*:

It is immediate that $\mathcal{B}$ is a subspace of $\mathbb{C}$. Recall that for each $k \in \mathbb{Z}$, $\delta_k \in \mathcal{A}$ is defined by $\delta_k(j) = 1$ if $j = k$, while $\delta_k(j) = 0$ otherwise. Then it is clear that $f = \sum_{k\in\mathbb{Z}} f(k)\delta_k$ (w.r.t. $\|\cdot\|_1$) for all $f \in \mathcal{A}$.

Hence, if $f \in \mathcal{B}$, we get that $f = \sum_{k=0}^{\infty} f(k)\delta_k$ (w.r.t. $\|\cdot\|_1$). It follows readily that

$$\mathcal{B} = \overline{\mathrm{span}\{\delta_k : k \in \mathbb{N} \cup \{0\}\}}^{\|\cdot\|_1},$$

so $\mathcal{B}$ is closed in $\mathcal{A}$. Moreover, as $\delta_k * \delta_l = \delta_{k+l}$ for all k, l, it is also straightforward to deduce that $\mathcal{B}$ is closed under the convolution product. Finally, the unit of $\mathcal{A}$ is δ_0, which obviously belongs to $\mathcal{B}$.

b) *Let* $\lambda \in \mathbb{D} := \{z \in \mathbb{C} : |z| \leq 1\}$. *Then the map* $\gamma_\lambda : \mathcal{B} \to \mathbb{C}$ *given by*

$$\gamma_\lambda(f) = \sum_{n=0}^{\infty} f(n)\lambda^n$$

for all $f \in \mathcal{B}$ *is well-defined and belongs to* $\widehat{\mathcal{B}}$:

Let $f \in \mathcal{B}$. Since $\sum_{n=0}^{\infty} |f(n)\lambda^n| \leq \sum_{n=0}^{\infty} |f(n)| = \|f\|_1 < \infty$, the complex series $\sum_{n=0}^{\infty} f(n)\lambda^n$ is absolutely convergent, hence convergent. Thus $\gamma_\lambda(f)$ makes sense, so we get that $\gamma_\lambda : \mathcal{B} \to \mathbb{C}$ is well-defined.

We have now to check that γ_λ is linear, bounded, multiplicative and nonzero. Linearity is easy. Since

$$|\gamma_\lambda(f)| \leq \sum_{n=0}^{\infty} |f(n)| = \|f\|_1$$

for all $f \in \mathcal{B}$, γ_λ is bounded. To show multiplicativity, consider first $k, l \in \mathbb{N} \cup \{0\}$. Then

$$\gamma_\lambda(\delta_k * \delta_l) = \gamma_\lambda(\delta_{k+l}) = \lambda^{k+l} = \lambda^k \lambda^l = \gamma_\lambda(\delta_k)\gamma_\lambda(\delta_l).$$

It follows that γ_λ is multiplicative on the subalgebra $\mathcal{B}_0 := \text{span}\{\delta_k : k \in \mathbb{N}\}$. By density of $\mathcal{B}_0$ in $\mathcal{B}$ and continuity of γ_λ, we get that γ_λ is multiplicative on $\mathcal{B}$. Finally, $\gamma_\lambda(\delta_0) = \lambda^0 = 1$, so $\gamma_\lambda \neq 0$.

c) *The map $\lambda \mapsto \gamma_\lambda$ is a homeomorphism from $\mathbb{D}$ onto $\widehat{\mathcal{B}}$:*

- Injectivity: Assume $\gamma_\lambda = \gamma_\mu$ for $\lambda, \mu \in \mathbb{D}$. Then $\lambda = \gamma_\lambda(\delta_1) = \gamma_\mu(\delta_1) = \mu$.

- Surjectivity: Let $\gamma \in \widehat{\mathcal{B}}$. Set $\lambda = \gamma(\delta_1) \in \mathbb{C}$. Then $|\lambda| = |\gamma(\delta_1)| \leq \|\gamma\| \, \|\delta_1\|_1 = 1$. Sorry o $\lambda \in \mathbb{D}$. Further, since γ is continuous and $\delta_n = (\delta_1)^n$ for all n, we get that

$$\gamma(f) = \sum_{n=0}^{\infty} f(n)\gamma(\delta_n) = \sum_{n=0}^{\infty} f(n)\gamma(\delta_1)^n = \sum_{n=0}^{\infty} f(n)\lambda^n = \gamma_\lambda(f)$$

for all $f \in \mathcal{B}$, so $\gamma = \gamma_\lambda$.

- We have shown that the map $\lambda \mapsto \gamma_\lambda$ is a bijection ϕ from $\mathbb{D}$ onto $\widehat{\mathcal{B}}$, with inverse $\psi : \widehat{\mathcal{B}} \to \mathbb{D}$ given by $\gamma \mapsto \gamma(\delta_1)$. If $\{\gamma_\alpha\}$ is a net in $\widehat{\mathcal{B}}$ converging to $\gamma \in \widehat{\mathcal{B}}$, then $\psi(\gamma_\alpha) = \gamma_\alpha(\delta_1) \to \gamma(\delta_1) = \psi(\gamma)$. Thus, ψ is continuous. Since $\widehat{\mathcal{B}}$ is compact and $\mathbb{D}$ is Hausdorff, we get that $\phi = \psi^{-1}$ is continuous too. Hence ϕ is an homeomorphism, as desired.

Next, identifying $\widehat{\mathcal{B}}$ with $\mathbb{D}$, we get that the Gelfand transform of $\mathcal{B}$ is the map $\Gamma : \mathcal{B} \to C(\mathbb{D})$ given by

$$[\Gamma(f)](\lambda) = \sum_{n=0}^{\infty} f(n)\lambda^n \quad \text{for } f \in \mathcal{B} \text{ and } \lambda \in \mathbb{D}$$

Indeed, for all $f \in \mathcal{B}$ and $\lambda \in \mathbb{D}$, we get that

$$\Gamma(f)](\gamma_\lambda) = \gamma_\lambda(f) = \sum_{n=0}^{\infty} f(n)\lambda^n.$$

Thus, by identifying λ with γ_λ, we obtain the formula above.

It follows that Γ is one-to-one: Indeed, if $\Gamma(f_1) = \Gamma(f_2)$ for $f_1, f_2 \in \mathcal{B}$, then we get that

$$\sum_{n=0}^{\infty} f_1(n)\lambda^n = \sum_{n=0}^{\infty} f_2(n)\lambda^n$$

for all $\lambda \in \mathbb{D}$. By the uniqueness of a power series expansion around zero, we get that $f_1(n) = f_2(n)$ for all $n \geq 0$, i.e., $f_1 = f_2$.

The range of Γ may be decribed as the set of continuous complex functions on $\mathbb{D}$ having an absolutely convergent power series expansion around zero valid in $\mathbb{D}$, i.e., which are analytic in $\mathbb{D}^o$. It follows that $\Gamma(\mathcal{B})$ is the disk algebra $\mathcal{A}(\mathbb{D})$.