MAT4450 - Spring 2024 - Solutions of exercises - Set 11

Exercise 45

Assume Ω is a non-compact, locally compact Hausdorff space. Consider the set of complex continuous functions on Ω given by $\mathcal{A} = C_0(\Omega) + \mathbb{C}\,1_\Omega$.

a) *$\mathcal{A}$ is a unital commutative C^*-algebra w.r.t. $\|\cdot\|_\infty$:*

Let $\mathcal{B} := C_b(\Omega)$ denote the commutative unital C*-algebra consisting of all complex bounded continuous functions on Ω, equipped with the $\|\cdot\|_\infty$-norm. Since $C_0(\Omega)$ is a closed subspace of $\mathcal{B}$, and $\mathbb{C}\,1_\Omega$ is a finite-dimensional subspace of $\mathcal{B}$, $\mathcal{A} = C_0(\Omega) + \mathbb{C}\,1_\Omega$ is a closed subspace of $\mathcal{B}$. It is elementary to check that $\mathcal{A}$ is closed under multiplication and that it is self-adjoint. Since $1_\Omega \in \mathcal{A}$, the assertion follows.

b) *$\widehat{\mathcal{A}}$ may be identified with the one-point compactification $\Omega \cup \{\infty\}$ of Ω:*

Let $\gamma \in \widehat{\mathcal{A}}$. Denote by γ^0 the restriction of γ to $C_0(\Omega)$. Since $\gamma(1_\Omega) = 1$, we get that

$$\gamma(f + \lambda 1_\Omega) = \gamma^0(f) + \lambda$$

for $f \in C_0(\Omega)$ and $\lambda \in \mathbb{C}$. Letting $\widetilde{\infty} : \mathcal{A} \to \mathbb{C}$ be given by $\widetilde{\infty}(f + \lambda 1_\Omega) := \lambda$ for $f \in C_0(\Omega)$ and $\lambda \in \mathbb{C}$, we see that $\widetilde{\infty} \in \widehat{\mathcal{A}}$ and

$$\widehat{\mathcal{A}} = \{\gamma \in \widehat{\mathcal{A}} \mid \gamma^0 \neq 0\} \cup \{\widetilde{\infty}\} \quad \text{(disjoint union).}$$

Now, for $\omega \in \Omega$, it is easy to check that $\gamma_\omega : \mathcal{A} \to \mathbb{C}$ defined by $\gamma_\omega(g) := g(\omega)$ for all $g \in \mathcal{A}$ belongs to $\widehat{\mathcal{A}}$ and satisfies $(\gamma_\omega)^0 \neq 0$. One verifies then without too much trouble that the map $\phi : \Omega \cup \{\infty\} \to \widehat{\mathcal{A}}$ defined by

$$\phi(t) := \begin{cases} \gamma_\omega & \text{if } t = \omega \in \Omega, \\ \widetilde{\infty} & \text{if } t = \infty, \end{cases}$$

is a homeomorphism between these two compact Hausdorff spaces.

It follows that $\mathcal{A}$ is isometrically $*$-isomorphic to $C(\Omega \cup \{\infty\})$:

Let $\Phi : C(\widehat{\mathcal{A}}) \to C(\Omega \cup \{\infty\})$ be defined by $\Phi(F) := F \circ \phi$ for all $F \in C(\widehat{\mathcal{A}})$. Using that ϕ is a homeomorphism, it is straightforward to check directly that Φ is an isometric $*$-isomorphism (you can also use the next exercise if you want). As $\mathcal{A}$ is isometrically $*$-isomorphic to $C(\widehat{\mathcal{A}})$ by Gelfand's theorem, the desired conclusion follows.

Exercise 46

Let Ω, Ω' be topological spaces. Recall that $C_b(\Omega)$ denotes the commutative unital C^*-algebra consisting of all complex bounded continuous functions on Ω, equipped with the $\|\cdot\|_\infty$-norm. We denote the character space $\widehat{C_b(\Omega)}$ of $C_b(\Omega)$ by $\beta\Omega$. By Gelfand's theorem, $\beta\Omega$ is a compact Hausdorff space.

a) *For $\omega \in \Omega$, define $\iota_\omega : C_b(\Omega) \to \mathbb{C}$ by $\iota_\omega(f) = f(\omega)$ for all $f \in C_b(\Omega)$. Then $\iota_\omega \in \beta\Omega$, and the map $\omega \mapsto \iota_\omega$ from Ω into $\beta\Omega$ is continuous*:

It is obvious that ι_ω is linear, multiplicative and maps 1_Ω to 1, hence that it belongs to $\beta\Omega$. If $\{\omega_\alpha\}$ is a net in Ω converging to $\omega \in \Omega$, then $\iota_{\omega_\alpha}(f) = f(\omega_\alpha) \to f(\omega) = \iota_\omega(f)$ for all $f \in C_b(\Omega)$. Thus ι_{ω_α} converges to ι_ω in $\widehat{C_b(\Omega)} = \beta\Omega$. This shows that $\omega \mapsto \iota_\omega$ is continuous.

b) *Let $h : \Omega \to \Omega'$ be a continuous map. Define a map $\Phi_h : C_b(\Omega') \to C_b(\Omega)$ by*

$$\Phi_h(g) = g \circ h$$

for each $g \in C_b(\Omega')$.

i) Φ_h *is a bounded* $*$*-homomorphism satisfying* $\|\Phi_h\| = 1$:

One checks readily that Φ_h is linear, multiplicative, and adjoint-preserving. For example, we have $\Phi_h(\bar{g}) = \bar{g} \circ h = \overline{g \circ h} = \overline{\Phi_h(g)}$ for all $g \in C(\Omega')$. Moreover, we have $\|\Phi_h(g)\|_\infty = \|g \circ h\|_\infty \leq \|g\|_\infty$ for all $g \in C(\Omega')$, so Φ_h is bounded with $\|\Phi_h\| \leq 1$. Since $\|\Phi_h(1_{\Omega'})\|_\infty = \| \, 1_\Omega\|_\infty = 1$ and $\|1_{\Omega'}\|_\infty = 1$, we also have $\|\Phi_h\| \geq 1$. Hence, $\|\Phi_h\| = 1$.

ii) Φ_h *is isometric (and therefore injective) when* h *is surjective*:

Assume h is surjective. Let $g \in C_b(\Omega')$. Then

$$\|\Phi_h(g)\|_\infty = \sup\{\big|g\big(h(\omega')\big)\big| : \omega' \in \Omega'\} = \sup\{\big|g(\omega)\big| : \omega \in \Omega\} = \|g\|_\infty.$$

iii) *Assume that* Ω, Ω' *are both compact Hausdorff spaces. Using Tietze's extension theorem, we get that* Φ_h *is surjective whenever* h *is injective*:

We first note that $h(\Omega)$ is a compact subset of Ω'. Hence it also closed (since Ω' is Hausdorff). Now, assume that h is injective and let $f \in C_b(\Omega)$. The map $h : \Omega \to h(\Omega)$ is a continuous bijection, hence a homeomorphism. Letting $k : h(\Omega) \to \Omega$ denote its inverse, we have that $f \circ k : h(\Omega) \to \mathbb{C}$ is continuous. By Tietze's extension theorem, we can extend $f \circ k$ to some $g \in C(\Omega')$. This gives that

$$\Phi_h(g) = g \circ h = f \circ k \circ h = f \circ \mathrm{id}_\Omega = f.$$

This shows that $\Phi_h\big(C(\Omega')\big) = C(\Omega)$, as desired.

It follows that Φ_h *is an isometric* $*$*-isomorphism from* $C(\Omega')$ *onto* $C(\Omega)$ *whenever* h *is a homeomorphism*:

Assume h is a homeomorphism. Using i) and ii) we get that Φ_h is a $*$-homomorphism which is isometric (hence injective). As seen above, Φ_h is also surjective. So Φ_h is an isometric $*$-isomorphism.

c) *Assume* $\Phi : C_b(\Omega') \to C_b(\Omega)$ *is an (algebra-)homomorphism such that* $\Phi(1_{\Omega'}) = 1_\Omega$. *Define* $H_\Phi : \beta\Omega \to \beta\Omega'$ *by*

$$H_\Phi(\gamma) = \gamma \circ \Phi \quad \text{for all } \gamma \in \beta\Omega.$$

Then H_Φ *is well-defined and continuous*:

Let $\gamma \in \beta\Omega$. Then $\gamma \circ \Phi : C_b(\Omega') \to C$ is linear and multiplicative. Moreover, $(\gamma \circ \Phi)(1_{\Omega'}) = \gamma(1_\Omega) = 1$. So $\gamma \circ \Phi$ is a character on $C_b(\Omega')$. Thus $H_\Phi(\gamma) = \gamma \circ \Phi \in \beta\Omega'$, showing that the map H_Φ takes its values in $\beta\Omega'$, as asserted in its definition.
Next, assume that $\{\gamma_\alpha\}$ is a net in $\beta\Omega$ converging to some $\gamma \in \beta\Omega$. Then for every $g \in C_b(\Omega')$ we have

$$[H_\Phi(\gamma_\alpha)](g) = [\gamma_\alpha \circ \Phi](g) = \gamma_\alpha\big(\Phi(g)\big) \longrightarrow \gamma\big(\Phi(g)\big) = [H_\Phi(\gamma)](g).$$

Thus, $H_\Phi(\gamma_\alpha) \to H_\Phi(\gamma)$. This shows that H_Φ is continuous on $\beta\Omega$.

d) *The space* $\beta\Omega$ *satisfies the following universal property:*
For any continuous function $h : \Omega \to K$ *from* Ω *into some compact Hausdorff space* K, *there exists a continuous function* $\tilde{h} : \beta\Omega \to K$ *such that*

$$\tilde{h}(\iota_\omega) = h(\omega) \quad \text{for all } \omega \in \Omega.$$

Let K be a compact Hausdorff space and assume $h : \Omega \to K$ is continuous. Using b) we can form the (algebra-) homomorphism $\Phi_h : C_b(K) = C(K) \to C_b(\Omega)$. Since it maps 1_K to 1_Ω, we can use

c) and form the continuous map $H_{\Phi_h} : \beta\Omega \to \beta K$. Now, recall that the map $k \mapsto \delta_k$ from K into $\widehat{C(K)} = \beta K$, where $\delta_k(g) := g(k)$ for $k \in K$ and $g \in C(K)$, is an homeomorphism.
Let $\omega \in \Omega$. Then for all $g \in C(K)$ we have

$$(\iota_\omega \circ \Phi_h)(g) = \iota_\omega(\Phi_h(g)) = [\Phi_h(g)](\omega) = (g \circ h)(\omega) = g(h(\omega)) = \delta_{h(\omega)}(g).$$

Thus,

$$H_{\Phi_h}(\iota_\omega) = \iota_\omega \circ \Phi_h = \delta_{h(\omega)}.$$

Let $\delta^{-1} : \beta K \to K$ denote the inverse of the homeomorphism given by $k \mapsto \delta_k$. Then the map $\tilde{h} := \delta^{-1} \circ H_{\Phi_h}$ is a continuous map from $\beta\Omega$ into K satisfying

$$\tilde{h}(\iota_\omega) = \delta^{-1}\big(H_{\Phi_h}(\iota_\omega)\big) = \delta^{-1}(\delta_{h(\omega)}) = h(\omega) \quad \text{for all } \omega \in \Omega,$$

as desired.

e) *Assume that Ω, Ω' are both compact Hausdorff spaces. Then the following assertions are equivalent*:

(i) *Ω and Ω' are homeomorphic.*

(ii) *$C(\Omega)$ and $C(\Omega')$ are isometrically $*$-isomorphic.*

(iii) *$C(\Omega)$ and $C(\Omega')$ are isomorphic.*

The implication (i) $\Rightarrow$ (ii) follows from the last part of b), while the implication (ii) $\Rightarrow$ (iii) is trivial.
Assume that (iii) holds, i.e., there exists an isomorphism $\Phi : C_b(\Omega') = C(\Omega') \to C_b(\Omega) = C(\Omega)$. Then c) gives us a continuous map H_Φ from $\beta\Omega$ to $\beta\Omega'$, which is easily seen to be an homeomorphism (with inverse map given by $H_{\Phi^{-1}}$). As $\beta\Omega$ (resp. $\beta\Omega'$) is homeomorphic to Ω (resp. Ω'), it follows that Ω and Ω' are homeomorphic, i.e., (i) holds. Thus, (iii) $\Rightarrow$ (i).

Exercise 47

b) Assume that $f \in C(\mathrm{sp}(D))$. Show that $f(D) \in \mathcal{B}(H)$ is the "diagonal" operator satisfying that $f(D)(e_j) = f(\lambda_j)\, e_j$ for every $j \in J$.

Let H be a nontrivial complex Hilbert space and $\mathcal{B} = \{e_j\}_{j\in J}$ be an orthonormal basis for H. Pick $\lambda_j \in \mathbb{C}$ for each $j \in J$, and assume that $\sup_{j\in J} |\lambda_j| < \infty$.

Let $D \in \mathcal{B}(H)$ denote the associated "diagonal" operator satisfying that $D(e_j) = \lambda_j\, e_j$ for every $j \in J$.

a) *D is normal* :
As seen in a previous course, D^* is the "diagonal" operator on H satisfying that $D^*(e_j) = \overline{\lambda_j}\, e_j$ for every $j \in J$. This implies that

$$(D^*D)(e_j) = \overline{\lambda_j}\lambda_j e_j = \lambda_j\overline{\lambda_j} e_j = (DD^*)(e_j) \quad \text{for each } j \in J.$$

Hence, $D^*D = DD^*$ (since a bounded operator is determined by its values on a orthonormal basis).

b) *Assume that $f \in C(\mathrm{sp}(D))$. Then $f(D) \in \mathcal{B}(H)$ is the "diagonal" operator satisfying that $f(D)(e_j) = f(\lambda_j)\, e_j$ for every $j \in J$*:

We recall that $\mathrm{sp}(D) = \overline{\{\lambda_j : j \in J\}}$. Let $f \in C(\mathrm{sp}(D))$. Then $\{f(\lambda_j) : j \in J\}$ is a bounded subset of $\mathbb{C}$ since $f(\mathrm{sp}(D))$ is compact. So we may form the diagonal operator $\psi(f)$ determined by $(\psi(f))(e_j) = f(\lambda_j)e_j$ for all $j \in J$. It is now straightforward to verify that the map $f \mapsto \psi(f)$ is a $*$-homomorphism from $C(\mathrm{sp}(D))$ into $\mathcal{B}(H)$ satisfying that $\psi\big(1_{\mathrm{sp}(D)}\big) = I_H$ and $\psi(\mathrm{id}) = D$, where $\mathrm{id}(z) = z$ for $z \in \mathbb{C}$. By the uniqueness property of the continuous functional calculus for D, we can conclude that $\psi(f) = f(D)$ for every $f \in C(\mathrm{sp}(D))$. This shows the assertion.