MAT4450 - Spring 2024 - Solutions of exercises - Set 12

Exercise 48

There were sorrily some misprints in the text of this exercise. The original idea was to assume that the Hilbert space H was finite-dimensional with $\dim(H) = n$, so that the spectrum of A was necessarily a finite set. However, one may skip the assumption that H is finite-dimensional and just assume that the spectrum of A is finite, as we do below.

Let H be a complex Hilbert space and consider $\mathcal{A} := \mathcal{B}(H)$ as a C^*-algebra with unit I (= the identity operator). Assume that $A \in \mathcal{A}$ is normal and has a finite spectrum, i.e., $\mathrm{sp}(A) = \{\lambda_1, \dots, \lambda_k\}$ where $k \in \mathbb{N}$ and $\lambda_j \neq \lambda_{j'}$ for $j \neq j'$. For each j, let $f_j : \mathrm{sp}(A) \to \mathbb{C}$ denote the indicator function of the set $\{\lambda_j\}$ inside $\mathrm{sp}(A)$, and set $P_j := f_j(A) \in \mathcal{A}$. We have seen that the P_j's are orthogonal projections in $\mathcal{A}$ which are are orthogonal to each other (i.e., $P_j P_j' = 0$ for $j \neq j'$) and satisfy

$$\sum_{j=1}^{k} P_j = I \quad \text{and} \quad \sum_{j=1}^{k} \lambda_j P_j = A.$$

For each j, let $E^A_{\lambda_j}$ denote the eigenspace of A corresponding to the eigenvalue λ_j, i.e., $E^A_{\lambda_j} = \ker\big(\lambda_j I - A\big)$.

Each P_j is the orthogonal projection from H onto $E^A_{\lambda_j}$:

We note that each λ_j is an isolated point of $\mathrm{sp}(A)$, so the claim follows from a result we proved in a lecture. We give a self-contained, elementary argument below.

For each j, set $M_j = P_j(H)$. Then each M_j is a closed subspace of H, and P_j is the orthogonal projection from H onto M_j. So we have to show that $M_j = E^A_{\lambda_j}$ for each j.

Assume first that $x \in M_j$. Then $P_j\, x = x$. On the other hand, if $j' \neq j$, then $P_j' P_j = 0$, so $P_{j'}\, x = P_{j'} P_j\, x = 0$. Hence we get that

$$Ax = \sum_{j'=1}^{k} \lambda_{j'} P_{j'} x = \lambda_j x\,,$$

i.e., $x \in E^A_{\lambda_j}$. Conversely, assume that $x \in E^A_{\lambda_j}$, i.e., $(\lambda_j I - A)x = 0$. Note that

$$\lambda_j I - A = \sum_{j'=1}^{k} \lambda_j P_{j'} - \sum_{j'=1}^{k} \lambda_{j'} P_{j'} = \sum_{j'=1}^{k} (\lambda_j - \lambda_{j'}) P_{j'} = \sum_{j' \in \{1,\dots,k\}, j' \neq j} (\lambda_j - \lambda_{j'}) P_{j'}.$$

Now, set

$$B := \sum_{l \in \{1,\dots,k\}, \ell \neq j} \frac{1}{\lambda_j - \lambda_l}\, P_\ell\,.$$

Using that $P_{j'} P_\ell = 0$ whenever $j' \neq \ell$, we get that

$$\begin{aligned} B\,(\lambda_j I - A) &= \Big(\sum_{l \in \{1,\dots,k\}, \ell \neq j} \frac{1}{\lambda_j - \lambda_l}\, P_\ell\Big)\Big(\sum_{j' \in \{1,\dots,k\}, j' \neq j} (\lambda_j - \lambda_{j'}) P_{j'}\Big) \\ &= \sum_{j' \in \{1,\dots,k\}, j' \neq j} P_{j'} = I - P_j\,. \end{aligned}$$

This gives that $(I - P_j)x = B(\lambda_j I - A)x = B\,0 = 0$, hence that $x = P_j x \in M_j$.

Exercise 49

Let H be a complex Hilbert space and consider $\mathcal{A} = \mathcal{B}(H)$ as a C^*-algebra.

Set $\mathcal{A}^+ = \{A \in \mathcal{A} : A \geq 0\}$. We recall that $A \geq 0$ if and only if $\langle A\xi, \xi\rangle \geq 0$ for all $\xi \in H$.

a) *The following properties of* $\mathcal{A}^+$ *hold.*

- $\mathcal{A}^+$ *is closed in* $\mathcal{A}$:

  Assume $\{A_n\}$ is a sequence in $\mathcal{A}^+$ converging to some $A \in \mathcal{A}$. Let $\xi \in H$. Then, using the Cauchy-Schwarz inequality, we get that $|\langle A_n\xi, \xi\rangle - \langle A\xi, \xi\rangle| \leq \|A - A_n\| \, \|\xi\|^2 \to 0$ as $n \to \infty$. Since $\langle A_n\xi, \xi\rangle \geq 0$ for every $n \in \mathbb{N}$, this gives that $\langle A\xi, \xi\rangle = \lim_{n\to\infty}\langle A_n\xi, \xi\rangle \geq 0$. Hence, $A \in \mathcal{A}^+$.

- *If* $A, B \in \mathcal{A}^+$ *and* $\lambda \in \mathbb{R}^+$, *then* $A + B \in \mathcal{A}^+$ *and* $\lambda A \in \mathcal{A}^+$: This is straightforward.

- *If* $A \in \mathcal{A}^+ \cap (-\mathcal{A}^+)$, *then* $A = 0$:

  Assume $A \in \mathcal{A}^+ \cap (-\mathcal{A}^+)$. Then we have that $\mathrm{sp}(A) \subseteq [0, \infty)$ and $\mathrm{sp}(A) \subseteq (-\infty, 0]$, i.e., $\mathrm{sp}(A) = \{0\}$. Since A is self-adjoint, hence normal, this implies that $\|A\| = r(A) = 0$, i.e., $A = 0$.

- *If* $A, B \in \mathcal{A}^+$ *and* $AB = BA$, *then* $AB \in \mathcal{A}^+$:

  Assume $A, B \in \mathcal{A}^+$ and $AB = BA$. Then $A^{1/2}B = BA^{1/2}$, so $AB = A^{1/2}A^{1/2}B = A^{1/2}BA^{1/2}$. Hence, for every $\xi \in H$, we have $\langle AB\xi, \xi\rangle = \langle A^{1/2}BA^{1/2}\xi, \xi\rangle = \langle BA^{1/2}\xi, A^{1/2}\xi\rangle \geq 0$. Thus, $AB \geq 0$.

b) *An example with* $H = \mathbb{C}^2$ *where* $A, B \in \mathcal{A}^+$, $AB \neq BA$ *and* $AB \notin \mathcal{A}^+$ *is as follows*:

Let $\{e_1, e_2\}$ denote the standard basis of $\mathbb{C}^2$. Let then $A, B \in \mathcal{B}(\mathbb{C}^2)$ be the operators satisfying $Ae_1 = e_1, Ae_2 = 2e_2, Be_1 = Be_2 = e_1 + e_2$. Then A and B are self-adjoint (look at their standard matrices). Moreover, $\mathrm{sp}(A) = \{1, 2\}$ and $\mathrm{sp}(B) = \{0, 2\}$ (since $B(e_1 - e_2) = 0$ and $B(e_1 + e_2) = 2(e_1 + e_2)$). Thus, $A, B \in \mathcal{A}^+$. As $ABe_1 = e_1 + 2e_2$, while $BAe_1 = e_1 + e_2$, we have $AB \neq BA$. Now, $(AB)^* = B^*A^* = BA \neq AB$, so AB is not self-adjoint. Thus, $AB \notin \mathcal{A}^+$.

Exercise 50

Let V be a complex vector space and $L : V \times V \to \mathbb{C}$ be a *sesquilinear form on* V, i.e., L is linear in the first variable and conjugate-linear in the second.

a) *We have*:

$$L(v, w) = \frac{1}{4}\sum_{k=0}^{3} i^k \, L\big(v + i^k w, v + i^k w\big) \quad \text{for all } v, w \in V.$$

(This is called the *polarization identity.*)

Indeed, a lengthy computation gives that

$$L\big(v + w, v + w\big) + iL\big(v + iw, v + iw\big) - L\big(v - w, v - w\big) - iL\big(v - iw, v - iw\big) = \cdots = 4L\big(v, w\big).$$

Hence, if L' is a sesquilinear form on V such that $L'(v, v) = L(v, v)$ for all $v \in V$, then $L' = L$, since the polarization identity gives that

$$L'(v, w) = \frac{1}{4}\sum_{k=0}^{3} i^k \, L'\big(v + i^k w, v + i^k w\big) = \frac{1}{4}\sum_{k=0}^{3} i^k \, L\big(v + i^k w, v + i^k w\big) = L(v, w)$$

for all $v, w \in V$.

b) Define $L^* : V \times V \to \mathbb{C}$ by $L^*(v,w) := \overline{L(w,v)}$ for all $v, w \in V$. *Then L^* is also a sesquilinear form on V:* This is straightforward.

It follows that L is self-adjoint*, that is, $L^* = L$, if and only if $L(v,v) \in \mathbb{R}$ for all $v \in V$:*

Assume $L^* = L$. Let $v \in V$. Then $L(v,v) = L^*(v,v) = \overline{L(v,v)}$, so $L(v,v) \in \mathbb{R}$. This shows the forward implication.
Conversely, assume $L(v,v) \in \mathbb{R}$ for all $v \in V$. Let $v, w \in V$. Then, using the polarization identity, we get that

$$L^*(v,w) = \frac{1}{4}\sum_{k=0}^{3} i^k\, L^*\big(v + i^k w, v + i^k w\big) = \frac{1}{4}\sum_{k=0}^{3} i^k\, \overline{L\big(v + i^k w, v + i^k w\big)}$$

$$= \frac{1}{4}\sum_{k=0}^{3} i^k\, L\big(v + i^k w, v + i^k w\big) = L(v,w).$$

c) Let H be a complex Hilbert space, $S \in \mathcal{B}(H)$ and let $L_S : H \times H \to \mathbb{C}$ denote the sesquilinear form on H given by $L_S(\xi,\eta) := \langle S(\xi), \eta\rangle$ for all $\xi, \eta \in H$.

We have that $(L_S)^* = L_{S^*}$:

Let $\xi, \eta \in H$. Then

$$(L_S)^*(\xi,\eta) = \overline{L_S(\eta,\xi)} = \overline{\langle S(\eta), \xi\rangle} = \langle \xi, S(\eta)\rangle = \langle S^*(\xi), \eta\rangle.$$

This shows the claim.

It follows that L_S is self-adjoint if and only if S is self-adjoint, if and only if $\langle S(\xi), \xi\rangle \in \mathbb{R}$ for all $\xi \in H$:

The first equivalence follows readily from the claim above. The second equivalence is an immediate consequence of the second statement in b) (applied to L_S).