MAT4450 - Spring 2024 - Solutions of exercises - Set 13

Exercise 51

Let H be a complex Hilbert space.

a) Assume that $T \in \mathcal{B}(H)$ is normal, $g \in B_b(\mathrm{sp}(T))$ and $f \in C(\Omega)$, where Ω is a compact subset of $\mathbb{C}$ containing $g(\mathrm{sp}(T))$.

Then we have $(f \circ g)(T) = f(g(T))$:

Set $\mathcal{P} := \{p : \Omega \to \mathbb{C} : p$ is a polynomial in z and $\overline{z}\}$. Then it is straightforward to check that $(p \circ g)(T) = p(g(T))$ for every $p \in \mathcal{P}$. Now it follows readily from the complex Stone-Weierstrass theorem that $\mathcal{P}$ is dense in $C(\Omega)$ (w.r.t. $\|\cdot\|_\infty$). We can therefore pick a sequence $\{p_n\}$ in $\mathcal{P}$ such that $p_n \to f$ as $n \to \infty$ (w.r.t. $\|\cdot\|_\infty$). Then $p_n(g(T)) \to f(g(T))$ as $n \to \infty$ (w.r.t. operator norm, hence also w.r.t. the SOT).

On the other hand, $\sup \|p_n\|_\infty < \infty$ (since $\{p_n\}$ is convergent w.r.t. $\|\cdot\|_\infty$), so $\{p_n \circ g\}$ is a bounded sequence of bounded Borel functions on $\mathrm{sp}(T)$ converging pointwise to $f \circ g$. Hence by the continuity property of the Borel functional calculus, we also get that $(p_n \circ g)(T) \to (f \circ g)(T)$ as $n \to \infty$ (w.r.t. the SOT).

As $p_n(g(T)) = (p_n \circ g)(T)$ for every $n \in \mathbb{N}$, we can conclude that the two limits are the same, i.e., $f(g(T)) = (f \circ g)(T)$, as desired.

b) Let $U \in \mathcal{B}(H)$ be unitary. *It follows, using a), that there exists a self-adjoint operator* $\Theta \in \mathcal{B}(H)$ *such that* $U = \exp(i\Theta)$:

Define $f : [0, 2\pi) \to \mathbb{T}$ by $f(t) = \exp(it)$. Then f is bijective and continuous, and its inverse function $h = f^{-1} : \mathbb{T} \to [0, 2\pi)$ is Borel measurable and bounded. Since $\mathrm{sp}(U)$ is a closed subset of $\mathbb{T}$, the function $g := h_{|\mathrm{sp}(U)}$ belongs to $B_b(\mathrm{sp}(U))$, so we can define $\Theta \in \mathcal{B}(H)$ by $\Theta = g(U)$. Since g is real-valued, Θ is self-adjoint. Using a), we get

$$U = \mathrm{id}(U) = (f \circ g)(U) = f(g(U)) = f(\Theta) = \exp(i\Theta).$$

Exercise 52

Let H be a complex Hilbert space and $T \in \mathcal{B}(H)$ be normal. Let $A \mapsto P(A)$ denote the projection-valued measure associated with T (so $P(A) := 1_A^{\mathrm{sp}(T)}(T)$ for every Borel subset A of $\mathrm{sp}(T)$). Moreover, let $\lambda \in \mathbb{C}$ and set $B(\lambda, \varepsilon) := \{\lambda' \in \mathbb{C} : |\lambda' - \lambda| < \varepsilon\}$.

a) *The following holds*:

$$\lambda \in \mathrm{sp}(T) \Leftrightarrow P\big(B(\lambda, \varepsilon) \cap \mathrm{sp}(T)\big) \neq 0 \text{ for all } \varepsilon > 0\,.$$

A more general result is true: see Proposition 4.5.10 in Pedersen's book. Choosing the function f to be equal to id on $\mathrm{sp}(T)$ in the proof of this result gives a proof of the statement above.

b) *Let* $\lambda \in \mathrm{sp}(T)$. *Then we have*:

$$\lambda \text{ is an eigenvalue of } T \Leftrightarrow P\big(\{\lambda\}\big) \neq 0,$$

in which case $P\big(\{\lambda\}\big)$ *is the orthogonal projection from* H *onto the eigenspace* $E_\lambda^T = \ker(\lambda I - T)$.

For a more general result, see the second part of Proposition 4.5.10 in Pedersen's book. You can find a direct proof of the statement above in Enstad's notes (see Proposition 4.7.3 (b) and its proof).

Exercise 53

Let H be a complex Hilbert space and $(X, \mathcal{M})$ be a measure space. Assume that $A \mapsto P(A)$ is a map from $\mathcal{M}$ into $\mathcal{B}(H)$ such that $P(A)$ is an orthogonal projection for every $A \in \mathcal{M}$, $P(\emptyset) = 0$, and $P(X) = I$. Consider the following conditions:

i) For every $\xi \in H$ and every sequence $\{A_j\}_{j \in \mathbb{N}}$ of pairwise disjoint sets in $\mathcal{M}$ we have that

$$P\Big(\bigcup_{j=1}^{\infty} A_j\Big)\,\xi = \sum_{j=1}^{\infty} P(A_j)\,\xi\,.$$

ii) For every $\xi, \eta \in H$ the map $\mu_{\xi,\eta} : \mathcal{M} \to \mathbb{C}$ defined by

$$\mu_{\xi,\eta}(A) = \big\langle P(A)\xi, \eta\big\rangle \quad \text{for all } A \in \mathcal{M}$$

is a complex measure on $(X, \mathcal{M})$.

a) *Assume that i) holds. Then ii) holds*:

Let $\xi, \eta \in H$. First, we have $\mu_{\xi,\eta}(\emptyset) = \big\langle P(\emptyset)\xi, \eta\big\rangle = \big\langle 0\,\xi, \eta\big\rangle = 0$. Next, let $\{A_j\}_{j \in \mathbb{N}}$ be a sequence of pairwise disjoint sets in $\mathcal{M}$. Then

$$\mu_{\xi,\eta}\Big(\bigcup_{j=1}^{\infty} A_j\Big) = \Big\langle P\Big(\bigcup_{j=1}^{\infty} A_j\Big)\xi, \eta\Big\rangle = \Big\langle \sum_{j=1}^{\infty} P(A_j)\,\xi, \eta\Big\rangle = \sum_{j=1}^{\infty} \big\langle P(A_j)\,\xi, \eta\big\rangle = \sum_{j=1}^{\infty} \mu_{\xi,\eta}(A_j).$$

Hence ii) holds.

Moreover, we have $\|\mu_{\xi,\eta}\| \leq \|\xi\|\,\|\eta\|$ for all $\xi, \eta \in H$:

Let $\xi, \eta \in H$. By definition, we have

$$\|\mu_{\xi,\eta}\| = |\mu_{\xi,\eta}|(\Omega) = \sup\Big\{\sum_{n=1}^{N} \big|\mu_{\xi,\eta}(E_n)\big|\Big\} = \sup\Big\{\sum_{n=1}^{N} \big|\langle P(E_n)\xi, \eta\rangle\big|\Big\}$$

where the sup's are taken over all finite measurable partitions $E_1, \ldots, E_N$ of Ω.

Consider such a partition $E_1, \ldots, E_N$. For each n, let $\lambda_n \in \mathbb{T}$ be such that

$$\big|\langle P(E_n)\xi, \eta\rangle\big| = \lambda_n\,\langle P(E_n)\xi, \eta\rangle.$$

Then we have

$$0 \leq \sum_{n=1}^{N} \big|\langle P(E_n)\xi, \eta\rangle\big| = \sum_{n=1}^{N} \lambda_n \langle P(E_n)\xi, \eta\rangle = \langle T\xi, \eta\rangle,$$

where $T := \sum_{n=1}^{N} \lambda_n\, P(E_n) \in \mathcal{B}(H)$. Note that

$$P(E_m)P(E_n) = P(E_m \cap E_n = P(\emptyset) = 0$$

for all $m \neq n$. Hence,

$$\|T\|^2 = \|T^*T\| = \Big\|\Big(\sum_{m=1}^{N} \lambda_m\, P(E_m)\Big)^*\Big(\sum_{n=1}^{N} \lambda_n\, P(E_n)\Big)\Big\| = \Big\|\sum_{m,n=1}^{N} \overline{\lambda_m}\lambda_n\, P(E_m)P(E_n)\Big\|$$

$$= \Big\|\sum_{n=1}^{N} P(E_n)\Big\| = \big\|P(\Omega)\big\| = \|I_H\| = 1.$$

So $\|T\| = 1$. Thus, using the Cauchy-Schwarz inequality, we get

$$\sum_{n=1}^{N} \big|\langle P(E_n)\xi, \eta\rangle\big| = \big|\langle T\xi, \eta\rangle\big| \leq \|T\|\,\|\xi\|\,\|\eta\| = \|\xi\|\,\|\eta\|.$$

Taking the sup over all finite measurable partitions $E_1, \ldots, E_N$ of Ω, we get

$$\|\mu_{\xi,\eta}\| \leq \|\xi\|\,\|\eta\|,$$

as claimed.

b) *Assume that i) holds. Then we have*:

$$P(A \cap B) = P(A)P(B) \quad \text{for all } A, B \in \mathcal{M}.$$

Let us first note that it follows readily from i) that P is finitely additive, that is,

$$P(E \cup F) = P(E) \cup P(F) \text{ whenever } E, F \in \mathcal{M} \text{ are disjoint.}$$

This implies that P is monotone, that is,

$$P(A) \leq P(B) \text{ whenever } A, B \in \mathcal{M} \text{ and } A \subseteq B,$$

since we then have

$$P(B) = P(A \cup (B \setminus A)) = P(A) + P(B \setminus A) \geq P(A).$$

We will also use that if Q, R are orthogonal projections in $\mathcal{B}(H)$, then $Q \leq R \Rightarrow RQ = Q = QR$ (the first equality holds because the range of Q is then contained in the range of R; the second follows by taking the adjoint).

Now let $A, B \in \mathcal{M}$. Then

$$P(A \cup B) = P(A \cup (B \setminus A)) = P(A) + P(B \setminus A)$$

$$\text{and} \qquad P(B) = P\big((B \setminus A) \cup (A \cap B)\big) = P(B \setminus A) + P(A \cap B).$$

$$\text{Thus} \qquad P(A \cup B) + P(A \cap B) = P(A) + P(B \setminus A) + P(A \cap B) = P(A) + P(B).$$

Multipliying with $P(A)$ from the left, we get

$$P(A)P(A \cup B) + P(A)P(A \cap B) = P(A) + P(A)P(B).$$

Since $A \subseteq (A \cup B)$, we have $P(A) \subseteq P(A \cup B)$, so $P(A)P(A \cup B) = P(A)$.

Similarly, since $(A \cap B) \subseteq A$, we have $P(A)P(A \cap B) = P(A \cap B)$.

Hence we get that $P(A) + P(A \cap B) = P(A) + P(A)P(B)$, i.e., $P(A \cap B) = P(A)P(B)$, as desired.