MAT4450 - Spring 2024 - Solutions of exercises - Set 14

Exercise 54

Let H be a Hilbert space (over $\mathbb{C}$) and $T \in \mathcal{B}(H)$. Let $T = U\,|T|$ be the polar decomposition of T.

a) *The following relations hold*:

- $U^*U|T| = |T|$
- $U^*T = |T|$
- $UU^*T = T$
- $|T^*| = U|T|U^*$
- $T^* = U^*|T^*|$, and this gives the polar decomposition of T^*.

We have seen in the lecture that U^*U is the orthogonal projection of H onto $\overline{|T|(H)}$, while UU^* is the orthogonal projection of H onto $\overline{T(H)}$.
This gives that

$$U^*U|T|\xi = |T|\xi \text{ and } UU^*T\xi = T\xi$$

for all $\xi \in H$, hence that $U^*U|T| = |T|$ and $UU^*T = T$. This implies that $U^*T = U^*U|T[= |T|$.
Moreover, since

$$|T^*|^2 = TT^* = U|T|(U|T|)^* = U|T|\,|T|U^* = U|T|\,U^*U|T|\,U^* = (U|T|U^*)^2,$$

and $U|T|U^* \geq 0$ (because $\langle U|T|U^*\xi, \xi\rangle = \langle |T|\,U^*\xi, U^*\xi\rangle \geq 0$ for all $\xi \in H$), we get that $|T^*| = U|T|U^*$ (by the uniqueness property of square roots). Finally, using this, we get

$$T^* = (U|T|)^* = |T|\,U^* = U^*U|T|U^* = U^*|T^*|.$$

Then U^* is a partial isometry. Further, since the final space of U is $\overline{T(H)}$, we have that $\ker(U^*) = U(H)^\perp = T(H)^\perp = \ker(T^*)$. So $T^* = U^*|T^*|$ is the polar decomposition of T^* (by the uniqueness property of polar decomposition).

b) Assume that T is invertible. Then U *is unitary*:

Since T is invertible, we have $\ker(T) = \{0\}$ and $T(H) = H$, hence $\overline{T(H)} = H$. So U^*U is the orthogonal projection of H onto

$$\overline{|T|(H)} = \ker(|T|)^\perp = \ker(T)^\perp = \{0\}^\perp = H,$$

that is, $U^*U = I_H$. Moreover, UU^* is the orthogonal projection of H onto $\overline{T(H)} = H$, i.e., $UU^* = I_H$. Thus, U is unitary.

Exercise 55

Let H be a Hilbert space (over $\mathbb{C}$) and let $T \in \mathcal{B}(H)$. *We describe the polar decomposition of T in the following cases*:

- $T \geq 0$

  We then have that $|T| = T$. Moreover, the partial isometry U restricts to the identity operator on $\overline{T(H)}$ and is zero on $\overline{T(H)}^\perp$. Hence U is the orthogonal projection of H onto $\overline{T(H)}$.

- *T is an orthogonal projection*

  This is a special case of the case above. We then get that $|T| = T$ and $U = T$.

- *T is a partial isometry*

  We then have that T^*T is an orthogonal projection. So $|T| = (T^*T)^{1/2} = T^*T$. Moreover, T^*T is the orthogonal projection of H onto $\overline{T(H)}$. Thus, $T = T\,T^*T = T\,|T|$, and this is the polar decomposition of T (by the uniqueness property), i.e., $U = T$

- *T is an isometry*

  This is a special case of the case above. We then get that $U = T$ and $|T| = I_H$.

Exercise 56

Assume H is a separable infinite-dimensional Hilbert space (over $\mathbb{C}$), and let $\{e_j : j \in \mathbb{N}\}$ be an o.n.b. for H. Let $S \in \mathcal{B}(H)$ denote the (unilateral) shift-operator associated with this basis, so S is determined by $S(e_j) = e_{j+1}$ for all $j \in \mathbb{N}$. Let $n \in \mathbb{N}$. *The following facts hold*:

- *S^n is an isometry with range $H_n := \{e_1, \ldots, e_n\}^\perp$*

  It should be well-known that S is an isometry, but here is the argument. Let $\xi \in H$. Then

  $$S(\xi) = S\Big(\sum_{j\in\mathbb{N}} \langle \xi, e_j\rangle\, e_j\Big) = \sum_{j\in\mathbb{N}} \langle \xi, e_j\rangle\, S(e_j) = \sum_{j\in\mathbb{N}} \langle \xi, e_j\rangle\, e_{j+1}.$$

  Since $\{e_{j+1}\}_{j\in\mathbb{N}}$ is an orthonormal set in H, we get that

  $$\|S(\xi)\|^2 = \Big\|\sum_{j\in\mathbb{N}} \langle \xi, e_j\rangle\, e_{j+1}\Big\|^2 = \sum_{j\in\mathbb{N}} \big|\langle \xi, e_j\rangle\big|^2 = \|\xi\|^2,$$

  which proves the claim.

  The product of two isometries is easily seen to be an isometry. It readily follows that S^n is an isometry.

  Moreover, using the formula for $S(\xi)$ above, we deduce that

  $$S^n(\xi) = \sum_{j\in\mathbb{N}} \langle \xi, e_j\rangle\, e_{j+n}.$$

  Thus, if H_n denotes the range of S^n, this implies that H_n is contained in $\overline{\mathrm{span}\{e_{j+n}, j \in \mathbb{N}\}}$. On the other hand, since $e_{j+n} = S(e_j)$ for each $j \in \mathbb{N}$, we get that $\mathrm{span}\{e_{j+n}, j \in \mathbb{N}\} \subseteq H_n$. As S^n is an isometry, its range H_n is complete, hence closed. Thus, we get that $\overline{\mathrm{span}\{e_{j+n}, j \in \mathbb{N}\}} \subseteq H_n$. Altogether, we conclude that $H_n = \overline{\mathrm{span}\{e_{j+n}, j \in \mathbb{N}\}}$.

  As $\{e_1, \ldots, e_n\}^\perp = \overline{\mathrm{span}\{e_{j+n}, j \in \mathbb{N}\}}$, this shows that $H_n = \{e_1, \ldots, e_n\}^\perp$, as desired.

- *$(S^n)^* = (S^*)^n$ is a partial isometry with initial space H_n and final space H*

  The operator S^n, being an isometry, is a partial isometry with initial space H and final space H_n. Thus, as seen in a lecture, this implies that $(S^n)^*$ is a partial isometry with initial space H_n and final space H.