MAT4450 - Spring 2024 - Solutions of exercises - Set 2

Exercise 6

Let X be a vector space (over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$) and let S be a nonempty family of seminorms on X. Let τ_S denote the weak topology on X induced by S. We recall that τ_S is the weak topology on X determined by the family $\{\rho_{(\sigma,z)}\}_{(\sigma,z)\in S\times X}$ of real-valued functions, where $\rho_{(\sigma,z)}(y) := \sigma(y-z)$ for all $y \in X$.

Let $x \in X$. For $\sigma \in S$ and $\varepsilon > 0$, set $B^\sigma_\varepsilon(x) := \{y \in X : \sigma(y-x) < \varepsilon\}$.

Moreover, for $z \in X$, set $V_{(\sigma,z),\,\varepsilon}(x) := \{y \in X : |\sigma(y-z) - \sigma(x-z)| < \varepsilon\}$.

a) *Let $z \in X$ and $\varepsilon > 0$. Then $V_{(\sigma,x),\,\varepsilon}(x) = B^\sigma_\varepsilon(x) \subseteq \; V_{(\sigma,z),\,\varepsilon}(x)$*:

The equality on the left follows immediately from the definitions. To show the inclusion on the right, let $y \in B^\sigma_\varepsilon(x)$. Then

$$|\sigma(y-z) - \sigma(x-z)| \leq \sigma\big((y-z)-(x-z)\big) = \sigma(x-y) < \varepsilon,$$

so $y \in V_{(\sigma,z),\,\varepsilon}(x)$, as desired.

b) If F is a finite nonempty subset of S and $\varepsilon > 0$, we set

$B^F_\varepsilon(x) := \bigcap_{\sigma\in F} B^\sigma_\varepsilon(x) = \{y \in X : \sigma(y-x) < \varepsilon \text{ for all } \sigma \in F\}$.

Then the family $\mathcal{B}_x := \big\{B^F_\varepsilon(x) : F$ is a finite nonempty subset of S and $\varepsilon > 0\big\}$ is a neighborhood basis at x (for τ_S):

Using Exercise 5 we get that the family

$$\mathcal{U}_x = \Big\{ \bigcap_{k=1}^{n} V_{(x_k,z_k),\,\varepsilon_k}(x) : \, n \in \mathbb{N}, x_1, z_1, \ldots, x_n, z_n \in X \text{ and } \varepsilon_1, \ldots, \varepsilon_n > 0 \Big\}$$

is a neighborhood basis at x (for τ_S). Using part a) we readily get that if $U \in \mathcal{U}_x$ (resp. $B \in \mathcal{B}_x$), then there exists $B \in \mathcal{B}_x$ (resp. $U \in \mathcal{U}_x$) such that $B \subseteq U$ (resp. $U \subseteq B$). The desired assertion clearly follows.

Exercise 7

Let $X = C(\mathbb{R},\mathbb{C})$ denote the vector space of all continuous complex functions on $\mathbb{R}$, and let $S = \{\sigma_K : K \subseteq \mathbb{R}, K \text{ compact}\}$ be the family of seminorms on X given by

$$\sigma_K(f) := \sup\{|f(t)| : t \in K\} \quad (f \in X)$$

for each $K \subseteq \mathbb{R}, K$ compact.

Then this family of seminorms is separating:

Let $f \in X, f \neq 0$. We have to show that there exists some $K \subseteq \mathbb{R}, K$ compact, such that $\sigma_K(f) \neq 0$. Pick $t_0 \in \mathbb{R}$ such that $f(t_0) \neq 0$, and set $\varepsilon := |f(t_0)|/2$. By continuity of the function $|f|$, we may find $\delta > 0$ such that

$$\big|\,|f(t)| - |f(t_0)|\,\big| < \varepsilon = |f(t_0)|/2 \quad \text{whenever } |t-t_0| \leq \delta.$$

This implies that $\varepsilon = \frac{|f(t_0)|}{2} < |f(t)|$ whenever $t \in K := [t_0-\delta, t_0+\delta]$. It follows immediately that $\sigma_K(f) > \varepsilon > 0$, as desired.

Exercise 8

Let X, X' be vector spaces (over the same field $\mathbb{F}$). Let S (resp. S') denote a family of seminorms on X (resp. X') and let τ (resp. τ') denote the weak topology on X (resp. X') induced by S (resp. S').

Let $T : X \to X'$ be a linear map, and consider T as a map from (X, τ) to (X', τ').
Then the following statements are equivalent:

a) *T is continuous on X*;

b) *T is continuous at* 0;

c) *For each $\sigma' \in S'$ there exist a (nonempty) finite subset F of S and $M > 0$ such that*

$$\sigma'\big(T(x)\big) \leq M \max_{\sigma \in F}\{\sigma(x)\} \quad \textit{for all } x \in X.$$

The implication a) $\Rightarrow$ b) is trivial. To show that b) $\Rightarrow$ c), assume that b) holds and let $\sigma' \in S'$. Set $U' := \sigma'^{-1}((-1,1)) = \{x' \in X' : \sigma'(x') < 1\}$, which is a τ'-open neigborhood of 0 in X'. Since $T(0) = 0$ and T is continuous at 0, there exists a τ-open neigborhood U of 0 in X such that $T(U) \subseteq U'$.

Using Exercise 6, we can find a finite nonempty subset F of S and $\varepsilon > 0$ such that $B^F_\varepsilon(0) \subseteq U$, i.e., $\{y \in X : \sigma(y) < \varepsilon \text{ for all } \sigma \in F\} \subseteq U$.

Set $M := 2/\varepsilon > 0$. Let $x \in X$ and set $m(x) := \max\{\sigma(x) : \sigma \in F\}$. To show that c) holds it suffices to check that $\sigma'\big(T(x)\big) \leq M\, m(x)$. There are two possibilities:

- $m(x) > 0$.

  In this case, we have that $\frac{1}{M\, m(x)}\, x = \frac{\varepsilon}{2\, m(x)}\, x \in B^F_\varepsilon(0) \subseteq U$. Hence we get that $T\big(\frac{1}{M\, m(x)}\, x\big) \in U'$, i.e., $\frac{1}{M\, m(x)}\, \sigma'(T(x)) < 1$, that is, $\sigma'\big(T(x)\big) \leq M\, m(x)$, as desired.

- $m(x) = 0$.

  This means that $\sigma(x) = 0$ for every $\sigma \in F$, and we have to deduce that $\sigma'(T(x)) = 0$. Let $\lambda \in \mathbb{F}$. Then we have that $\sigma(\lambda\, x) = |\lambda|\, \sigma(x) = 0 < \varepsilon$ for all $\sigma \in F$. Thus $\lambda\, x \in B^F_\varepsilon(0) \subseteq U$, so we get that $T(\lambda\, x) \in U'$. Hence $|\lambda|\, \sigma'(T(x)) = \sigma'(T(\lambda\, x)) < 1$. Since this is true for every $\lambda \in \mathbb{F}$, we must have that $\sigma'(T(x)) = 0$, as desired.

We have thereby shown that b) $\Rightarrow$ c).

Finally, assume that c) holds. Let $x \in X$ and let $\{x_\alpha\}$ be a net in X converging to x. To show that a) holds, it suffices to show that $T(x_\alpha) \to_\alpha T(x)$, i.e., $\sigma'\big(T(x_\alpha) - T(x)\big) \to_\alpha 0$ for every $\sigma' \in S'$, that is, $\sigma'\big(T(x_\alpha - x)\big) \to_\alpha 0$ for every $\sigma' \in S'$.

Let $\sigma' \in S'$ and $\varepsilon > 0$. Using the assumption, we may pick a (nonempty) finite subset F of S and $M > 0$ such that $\sigma'\big(T(y)\big) \leq M \max_{\sigma \in F}\{\sigma(y)\}$ for all $y \in X$. In particular, we have that

$$\sigma'\big(T(x_\alpha - x)\big) \leq M \max_{\sigma \in F}\{\sigma(x_\alpha - x)\} \text{ for all } \alpha.$$

We choose now α_0 such that $\sigma(x_\alpha - x) < \varepsilon/M$ for all $\sigma \in F$ and all $\alpha \gtrsim \alpha_0$. Then we clearly get that $\sigma'\big(T(x_\alpha - x)\big) < \varepsilon$ for all $\alpha \gtrsim \alpha_0$, which shows that $\sigma'\big(T(x_\alpha - x)\big) \to_\alpha 0$, as desired. Thus we have shown that c) $\Rightarrow$ a).

Exercise 9

Let X be topological vector space and let $\ell : X \to \mathbb{F}$ be a linear functional. Then the following conditions are equivalent:

a) *ℓ is continuous on X.*

b) *ℓ is continuous at some point of X.*

c) $\sup\{\operatorname{Re}\ell(u) \mid u \in U\} < \infty$ *for some nonempty open $U \subseteq X$.*

d) $\inf\{\operatorname{Re}\ell(u) \mid u \in U\} > -\infty$ *for some nonempty open $U \subseteq X$.*

e) $\sup\{|\ell(u)| \mid u \in U\} < \infty$ *for some nonempty open $U \subseteq X$.*

$a) \Rightarrow b)$: This implication is obvious.

$b) \Rightarrow c)$: Assume that ℓ is continuous at $x_0 \in X$. Then ℓ is continuous at $0 \in X$. Indeed, if $\{y_\alpha\}$ is a net in X converging to 0, then $(x_0 + y_\alpha) \to_\alpha x_0$, so $\ell(x_0) + \ell(y_\alpha) = \ell(x_0 + y_\alpha) \to_\alpha \ell(x_0)$. Hence $\ell(y_\alpha) \to_\alpha 0 = \ell(0)$, and the claim follows. This implies that $\operatorname{Re}\ell$ is continuous at 0. In particular, there exists an open $U \subseteq X$ such that $\operatorname{Re}\ell(U) \subseteq (-1, 1)$, and it clearly follows that $c)$ holds.

$c) \Rightarrow d)$: Assume that $c)$ holds. So there exists some $M \in \mathbb{R}$ and some nonempty open $U \subseteq X$ such that $\operatorname{Re}\ell(U) \subseteq (-\infty, M]$. Then we have $\operatorname{Re}\ell(-u) = -\operatorname{Re}\ell(u) \in [-M, \infty)$ for all $u \in U$. Thus, $V := -U$ is a nonempty open subset of X such that $\inf\{\operatorname{Re}\ell(v) \mid v \in V\} \geq -M > -\infty$, showing that $d)$ holds.

$d) \Rightarrow e)$: Assume that $d)$ holds. We may assume that $m := \inf\{\operatorname{Re}\ell(u) \mid u \in U\} > -\infty$ for some nonempty open neighborhood U of 0 (because if it happens that $0 \notin U$, then we just pick any $x_0 \in U$, and replace U by $U' = U - x_0$).

By continuity of multiplication by scalars, we may pick $\delta > 0$ and a nonempty open $V \subseteq X$ such that $\lambda v \in U$ for all $\lambda \in B_\delta(0)$ and all $v \in V$.

Let $v \in V$ and pick $r \in (0, \delta)$. Then for all $t \in \mathbb{R}$ we have $r\, e^{it} v \in U$. Thus we get

$$m \leq \operatorname{Re}\big(\ell(r\, e^{it}\, v)\big) = \operatorname{Re}\big(r\, e^{it}\, \ell(v)\big)$$

for all $t \in \mathbb{R}$. Choosing t such that $e^{it}\,\ell(v) = -|\ell(v)|$ gives $m \leq -r\,|\ell(v)|$, so $|\ell(v)| \leq -m/r$. Since this holds for every $v \in V$, we get

$$\sup\{\,|\ell(v)| \mid v \in V\} \leq -\frac{m}{r} < \infty\,,$$

so $e)$ holds.

$e) \Rightarrow a)$: Assume that e) holds. By translating U if necessary, we may assume that $M := \sup\{\,|\ell(u)| \mid u \in U\} < \infty$ for some nonempty open neighborhood U of 0.

Let $x \in X$. To show that ℓ is continuous at x, let $\{x_i\}$ be a net in X converging to x. Pick $0 < r < 1$ and an open neighborhood V of 0 such that $\lambda v \in U$ for all $\lambda \in B_r(0)$ and all $v \in V$. Let $\varepsilon > 0$. Set

$$W := \varepsilon \Big(\frac{r}{M + r + 1}\Big)^2 V\,.$$

Since W is an open neighborhood of 0, we may find i_0 such that $(x_i - x) \in W$ whenever $i \gtrsim i_0$.

Now, as $\frac{r}{M+r+1} < r$, we have that $\frac{r}{M+r+1} V \subseteq U$, which gives that $W \subseteq \frac{\varepsilon\, r}{M+r+1} U$.

Hence, we get that

$$(x_i - x) \in \frac{\varepsilon\, r}{M + r + 1}\, U \quad \text{whenever } i \gtrsim i_0\,.$$

Thus,

$$|\ell(x_i) - \ell(x)| \;=\; |\ell(x_i - x)| \;\le\; \frac{\varepsilon\, r}{M + r + 1}\, M < \varepsilon\, r < \varepsilon$$

whenever $i \gtrsim i_0$. This shows that $\ell(x_i) \to_\alpha \ell(x)$. Thus we have shown that ℓ is continuous at x. It follows that *a*) holds.

Exercise 10

Let X be an infinite-dimensional normed space.

a) *Let U be a weakly open neighborhood of* 0 *in* X *(i.e.,* $0 \in U \subseteq X$ *and* $U \in \tau_{\text{weak}}$*). Then U contains an infinite-dimensional subspace of X.*

If $U = X$, then the assertion is trivial. So we may assume that $U \neq X$. Exercise 6 gives that there exist $\varepsilon > 0$ and $\varphi_1, \ldots, \varphi_n \in X^*$ such that

$$\bigcap_{j=1}^{n} \{x \in X : |\varphi_j(x)| < \varepsilon\} \;\subseteq\; U\,.$$

In particular, we have $\bigcap_{j=1}^{n} \ker \varphi_j \;\subseteq\; U \neq X$. Now, consider the linear map $L : X \to \mathbb{F}^n$ defined by

$$L(x) = \big(\varphi_1(x), \ldots, \varphi_n(x)\big)$$

for all $x \in X$. Then $\ker L = \bigcap_{j=1}^{n} \ker \varphi_j \subseteq U$, and $L(X) \neq \{0\}$ (otherwise, all the φ_j's would be equal to the zero functional on X, so we would have $\ker L = X \subseteq U \neq X$, a contradiction).

We claim that the subspace $\ker L$ is infinite-dimensional:

Assume (for contradiction) that $\ker L$ is finite-dimensional. Since X is infinite-dimensional, we must have that $\ker L \neq \{0\}$. Pick a basis $v_1, \ldots, v_m$ for $\ker L$ and a basis $r_1, \ldots, r_p$ for $L(X)$ (so $1 \le p \le n$). Next, pick $u_1, \ldots, u_p \in X$ such that $L(u_k) = r_k$ for $k = 1, \ldots, p$. Then one easily checks that X is spanned by the vectors $u_1, \ldots, u_p, v_1, \ldots, v_m$. Hence X is finite-dimensional, which gives a contradiction. Thus, $\ker L$ must be infinite-dimensional.

As $\ker L$ is contained in U, the desired assertion is proven.

b) *The weak topology on X does not coincide with the norm topology.*

Assume (for contradiction) that these two topologies agree. Let U denote the open unit ball in X, i.e. $U = \{x \in X : \|x\| < 1\}$. Then the assumption gives that U is weakly open, so U contains an infinite-dimensional subspace of X, say M. Pick $v \in M$, $v \neq 0$. Then $\lambda\, v \in M$ for every $\lambda \in \mathbb{F}$, i.e., $|\lambda|\, \|v\| = \|\lambda\, v\| < 1$ for every $\lambda \in \mathbb{F}$. This implies that $|\lambda| < 1/\|v\|$ for every $\lambda \in \mathbb{F}$, which is impossible.