MAT4450 - Spring 2024 - Solutions of exercises - Set 3

Exercise 11

Let X be a normed space, $X \neq \{0\}$.

a) *Assume that X^* is finite-dimensional. Then X is finite-dimensional too*:

Indeed, we then have that $X^{**} = (X^*)^*$ is finite-dimensional. Let j denote the canonical map from X into X^{**} (given by $x \mapsto j_x$, where $j_x(\varphi) = \varphi(x)$). Since j is a linear isometry, and therefore injective, we get that $\dim(X) = \dim(j(X)) \leq \dim(X^{**}) < \infty$.

b) *X is finite-dimensional if and only if the weak*-topology on X^* agrees with the norm-topology on X^**:

Assume that X is finite-dimensional. Let j be as above. Then

$$\dim(j(X)) = \dim(X) = \dim(X^*) = \dim(X^{**}),$$

so $j(X) = X^{**}$. So the weak*-topology on X^*, being the weak topology on X^* determined by $j(X) = (X^*)^*$, is the same as the weak topology on the normed space X^*. Since X^* is finite-dimensional, we get that this topology agrees with the norm-topology on X^*.

Conversely, assume that the weak*-topology on X^* agrees with the norm-topology. Now, the weak topology on X^* is weaker than the norm topology, and stronger than the weak*-topology. So all these three topologies on X^* coincide, in particular, the weak topology on X^* agrees with the norm topology. This forces X^* to be finite-dimensional, which in turn forces X to be finite-dimensional by a). (Note: one could also have used d) for this part).

c) *X is finite-dimensional if and only if the closed unit ball B^* in X^* is compact (w.r.t. the norm-topology on X^*)*:

Using a) and known results, we get that

X is finite-dimensional $\Leftrightarrow$ X^* is finite-dimensional $\Leftrightarrow$ B^* is norm-compact in X^*.

d) *Assume that X is infinite-dimensional. Let W be a weak*-open neighborhood of 0 in X^*. Then W contains an infinite-dimensional subspace of X^**:

The proof is similar to the proof of Exercise 10 a), so we just sketch the proof. We may choose $x_1, \ldots, x_n \in X$ and $\varepsilon > 0$ such that

$$\bigcap_{j=1}^{n} \{\varphi \in X^* : |\varphi(x_j)| < \varepsilon\} \quad \subseteq W.$$

Consider now the linear map $L' : X^* \to \mathbb{F}^n$ given by $L'(\varphi) = \big(\varphi(x_1), \ldots, \varphi(x_n)\big)$ for all $\varphi \in X^*$. Clearly, we get that $\ker(L') \subseteq W$. Arguing essentially as in the proof of Exercice 10 a), one gets that $\ker(L')$ is infinite-dimensional, which shows the assertion.

Exercise 12

Let X be a vector space (over $\mathcal{F}$) and let A be a non-empty subset of X. We recall that $a \in A$ is called an internal point of A if for all $x \in X \setminus \{0\}$ there exists some $\varepsilon > 0$ such that $a + \lambda x \in A$ for all $\lambda \in \mathcal{F}$ satisfying $|\lambda| < \varepsilon$. We let A^{int} denote the set of all internal points of A.

a) *Assume that* X *is a topological vector space. Then* $A^o \subseteq A^{\text{int}}$:

Let $a \in A^o$ and $x \in X \setminus \{0\}$. We may then find $U \in \mathcal{N}_a$ such that $U \subseteq A$. By continuity of the map $\lambda \mapsto a + \lambda\, x$, we may then find $\varepsilon > 0$ such that $a + \lambda\, x \in U \subseteq A$ whenever $\lambda \in \mathbb{F}$, $|\lambda| < \varepsilon$. Hence, $a \in A^{\text{int}}$.

b) *Consider* $X = \mathbb{R}^2$ *with its usual topology, and* $A = \{(x, y) \in [-1, 1]^2 \mid x^2 \leq y \text{ or } y \leq 0\}$. *Then* $A^o \neq A^{\text{int}}$:

Indeed, making a drawing of A, one easily sees that $0 \in A^{\text{int}} \setminus A^o$.

c) *Assume that* X *is a finite-dimensional normed space and* A *is convex. Then* $A^o = A^{\text{int}}$:

We sketch a proof. Let $a \in A^{\text{int}}$. By replacing A with $A - a$ if necessary, we may assume that $a = 0$. We may then find a basis $\{x_1, \ldots, x_n\}$ for X such that $\lambda x_j \in A$ for every $\lambda \in T := \{\lambda \in \mathbb{F} : |\lambda| = 1\}$ and every $j \in \{1, \ldots, n\}$. Let $\|\cdot\|'$ be the norm on X given by

$$\|\sum_{j=1}^{n} c_j\, x_j\|' := \sum_{j=1}^{n} |c_j|$$

whenever $c_1, \ldots, c_n \in \mathbb{F}$. Now, if $x = \sum_{j=1}^{n} c_j\, x_j \in X$ is such that $\|x\|' < 1$, and we choose $\lambda_j \in T$ such that $c_j = \lambda_j\, |c_j|$ for every j, then x can be written as a convex combination of the vectors $0, \lambda_1\, x_1, \ldots, \lambda_n\, x_n$. As these vectors all belong to A, and A is convex, this implies that $\{x \in X : \|x\|' < 1\} \subseteq A$. But all norms on X are equivalent, so this shows that $a = 0$ is an interior point of A, as desired.

d) *Consider* $X = C([0, 1], \mathbb{R})$ *as a normed space (over* $\mathbb{R}$*) w.r.t. the norm* $\|f\|_1 = \int_0^1 |f(t)|\, dt$, *and* $A = \{f \in X \mid \sup_{t \in [0,1]} |f(t)| < 1\}$. *Then* A *is convex and* $A^o \neq A^{\text{int}}$:

Since A is the open unit ball in X w.r.t. to the sup-norm, A is convex. Moreover, 0 is an interior point of A w.r.t. to the sup-norm, so it is an internal point of A (by a)). However, 0 is not an interior point of A w.r.t. to $\|\cdot\|_1$. Indeed, let $\varepsilon > 0$ and consider $U_\varepsilon = \{f \in X : \|f\|_1 < \varepsilon\}$.

Assume first that $\varepsilon < 1$. Let then $g_\varepsilon \in X$ be the function given by $g_\varepsilon(t) = 1 - \varepsilon^{-1}\, t$ when $t \in [0, \varepsilon]$, and by $g_\varepsilon(t) = 0$ when $t \in [\varepsilon, 1]$. Then $\|g_\varepsilon\|_1 = \varepsilon/2 < 1$, so $g_\varepsilon \in U_\varepsilon$. But $\sup_{t \in [0,1]} |g_\varepsilon(t)| = 1$, so $g_\varepsilon \notin A$. Hence, $U_\varepsilon \not\subseteq A$.

Next, if $\varepsilon \geq 1$, then $g_{1/2} \in U_{1/2} \subseteq U_\varepsilon$, while $g_{1/2} \notin A$, so $U_\varepsilon \not\subseteq A$ in this case too.

This implies that no $\|\cdot\|_1$-neighborhood of 0 is contained in A, i.e., $0 \notin A^o$. Hence, $0 \in A^{\text{int}} \setminus A^o$, so $A^o \neq A^{\text{int}}$.