MAT4450 - Spring 2024 - Solutions of exercises - Set 4

Exercise 13

Here is an example of two nonempty disjoint (unbounded) closed convex subsets F and K of $\mathbb{R}^2$ where the separation property asserted in the Hahn-Banach separation theorem III is not satisfied:

Set $F := \{(x,y) \in \mathbb{R}^2 : y \geq e^x\}$ and $K := \{(x,y) \in \mathbb{R}^2 : y \leq 0\}$. Then there is no $\varphi \in (\mathbb{R}^2)^*$ such that

$$\sup\{\varphi(x,y) : (x,y) \in K\} < \inf\{\varphi(x,y) : (x,y) \in F\}.$$

This is easy to see by making a drawing, or by writing $(\mathbb{R}^2)^* = \{\varphi_{(a,b)} : (a,b) \in \mathbb{R}^2\}$, where $\varphi_{(a,b)}(x,y) := ax + by$, and studying the possible values of both sides of this inequality.
We note that the algebraic Hahn-Banach separation theorem guarantees that there exists a nonzero $\varphi \in (\mathbb{R}^2)^*$ such that $\sup\{\varphi(x,y) : (x,y) \in K\} \leq \inf\{\varphi(x,y) : (x,y) \in F\}$. This happens exactly when $\varphi = \varphi_{(0,b)}$, $b > 0$; both sides of the inequality are then equal to 0.

Exercise 14 (= Exercise 2.4.6 in Pedersen's book)

Let $\{x_n\}$ be a sequence in a normed space X, such that $\varphi(x_n) \to \varphi(x)$ for some $x \in X$ and all $\varphi \in X^*$. (This says that $x_n \to x$ in the weak topology of X). Let $\varepsilon > 0$ and $m \in \mathbb{N}$. Then there is a convex combination y of vectors in $X_m := \{x_n : n \geq m\}$ such that $\|x - y\| < \varepsilon$:

We first recall that if A is a nonempty subset of X, then $\text{co}(A)$ denotes the convex subset of X consisting of all convex combinations of vectors in A.

Set $S_m := \text{co}(X_m)$. Since S_m is a convex subset of X, we know that $\overline{S_m}^{\|\cdot\|} = \overline{S_m}^{\text{weak}}$. Since $\{x_n\}_{n \geq m}$ is a sequence in S_m converging weakly to x, this gives that $x \in \overline{S_m}^{\|\cdot\|}$. Thus, given $\varepsilon > 0$, there exists some $y \in S_m$ such that $\|x - y\| < \varepsilon$, as was to be shown.

Exercise 15

a) *Let $n, m \in \mathbb{N}$, and choose some norms on $\mathbb{F}^n$ and $\mathbb{F}^m$. Let A be a $m \times n$ matrix over $\mathbb{F}$, and $T : \mathbb{F}^n \to \mathbb{F}^m$ be the linear map having A as its standard matrix. We identify $\mathbb{F}^n$ with $(\mathbb{F}^n)^*$ via the map $y \mapsto \varphi_y$, where $\varphi_y(x) := x \cdot y = \sum_{i=1}^n x_i y_i$. Similarly, we identify $\mathbb{F}^m$ with $(\mathbb{F}^m)^*$.*

The standard matrix of the adjoint operator $T^* : \mathbb{F}^m \to \mathbb{F}^n$ is the transpose of A:

Let $y \in \mathbb{F}^m$. Then, for all $x \in \mathbb{F}^n$, we have

$$[T^*(y)](x) = (y \circ T)(x) = y(T(x)) = (Ax) \cdot y = x^t A^t y = x \cdot (A^t y) = [A^t y](x).$$

Thus $T^*(y) = A^t y$.

b) *Let X be a normed space over $\mathbb{F}$, and let $\varphi \in X^*$, i.e., $\varphi \in \mathcal{B}(X, \mathbb{F})$. Indentify $\mathbb{F}$ with $(\mathbb{F})^*$ in the obvious way, that is we consider $\lambda \in \mathbb{F}$ as the linear functional on $\mathbb{F}$ given by $\lambda(\mu) = \lambda\mu$ for all $\mu \in \mathbb{F}$.*

Then the adjoint map $\varphi^* \in \mathcal{B}(\mathbb{F}, X^*)$ is given by $\varphi^*(\lambda) = \lambda\varphi$ for all $\lambda \in \mathbb{F}$:

Let $\lambda \in \mathbb{F}$. Then, for all $x \in X$, we have

$$[\varphi^*(\lambda)](x) = [\lambda \circ \varphi](x) = \lambda(\varphi(x)) = \lambda\varphi(x) = [\lambda\varphi](x),$$

which proves the assertion.

Exercise 16

Consider $X = \ell^1(\mathbb{N}, \mathbb{F})$ as a normed space w.r.t. the $\|\cdot\|_1$-norm. Recall that $\ell^\infty(\mathbb{N}, \mathbb{F})$ (with the $\|.\|_\infty$-norm) may be identified with X^* via the isometric isomorphism $g \mapsto \varphi_g$, where $\varphi_g(f) := \sum_{n=1}^\infty f(n)g(n)$ for all $f \in \ell^1(\mathbb{N}, \mathbb{F})$ whenever $g \in \ell^\infty(\mathbb{N}, \mathbb{F})$.

Set $Y := c_0(\mathbb{N}, \mathbb{F}) = \{g \in \ell^\infty(\mathbb{N}, \mathbb{F}) \mid \lim_{n\to\infty} g(n) = 0\}$. Recall also that X may be identified with Y^* (when Y is equipped with the $\|.\|_\infty$-norm), via the isometric isomorphim $f \mapsto \psi_f$, where $\psi_f(g) = \sum_{n=1}^\infty f(n)g(n)$ for all $f \in X = \ell^1(\mathbb{N}, \mathbb{F})$ whenever $g \in Y = c_0(\mathbb{N}, \mathbb{F})$.

a) *Let $T : X \to Y$ be the linear map given by $[T(f)](n) = \sum_{m=n}^\infty f(m)$ for all $f \in X$ and $n \in \mathbb{N}$. Then T is bounded*:

Indeed, let $f \in X$. Then we have that

$$\big|[T(f)](n)\big| \le \sum_{m=n}^\infty |f(m)| \le \sum_{m=1}^\infty |f(m)| = \|f\|_1$$

for all $n \in \mathbb{N}$. So $\|T(f)\|_\infty \le \|f\|_1$. Hence, T is bounded, with $\|T\| \le 1$.

An expression for $T^* \in \mathcal{B}(Y^*, X^*) = \mathcal{B}(X, X^*)$ is as follows.

Let $f \in Y^* = X$. Then, for all $h \in X$, we get (using Fubini)

$$\begin{aligned}[T^*(f)](h) = (f \circ T)(h) = f(T(h)) = \sum_{n=1}^\infty f(n)[T(h)](n) &= \sum_{n=1}^\infty \sum_{m=n}^\infty f(n)h(m) \\ = \sum_{m=1}^\infty \sum_{n=1}^m h(m)f(n) = \sum_{m=1}^\infty h(m)g(m) &= [g](h),\end{aligned}$$

where $g \in X^* = \ell^\infty(\mathbb{N}, \mathbb{F})$ is given by $g(m) := \sum_{n=1}^m f(n)$ for all $m \in \mathbb{N}$.
Thus, $[T^*(f)](m) = \sum_{n=1}^m f(n)$ for all $m \in \mathbb{N}$.

b) *Consider Y as a subspace of X^*. Then Y is norm-closed in X^**:

Let $\{g_m\}$ be a sequence in Y such that $\|g_m - g\|_\infty \to 0$ as $m \to \infty$ for some $g \in X^* = \ell^\infty(\mathbb{N}, \mathbb{F})$. Then for all $n, m \in \mathbb{N}$ we have

$$|g(n)| \le |g(n) - g_m(n)| + |g_m(n)| \le \|g - g_m\|_\infty + [g_m(n)|$$

Let $\varepsilon > 0$. Then choose first $m \in \mathbb{N}$ such that $\|g - g_m\|_\infty < \varepsilon/2$. Then choose $N \in \mathbb{N}$ such that $|g_m(n)| < \varepsilon/2$ for all $n \ge N$. Then we get that $|g(n)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$ for all $n \ge N$. This shows that $g(n) \to 0$ as $n \to \infty$, i.e., $g \in Y$, as desired.

Moreover, we have that $(Y^\perp)^\perp = X^*$ (so $Y \ne (Y^\perp)^\perp$):

We first show that $Y^\perp = \{0\}$: Let $f \in Y^\perp$. So $f \in X$ and $\sum_{n=1}^\infty f(n)g(n) = 0$ for all $g \in Y$. Choosing g to be the indicator function of the set $\{m\}$ for any $m \in \mathbb{N}$ gives that $f(m) = 0$. Thus $f = 0$, as desired. Using this we get $(Y^\perp)^\perp = \{0\}^\perp = X^*$.