MAT4450 - Spring 2024 - Solutions of exercises - Set 5

Exercise 17

We consider $\ell^1 := \ell^1(\mathbb{N}, \mathbb{F})$ as a Banach space with the $\|\cdot\|_1$-norm, and the map $T : \ell^1 \to \ell^1$ defined by

$$[T(f)](n) = \frac{1}{n} f(n), \quad n \in \mathbb{N}$$

for all $f \in \ell^1$.

a) *T is well-defined, injective and bounded, with $\|T\| = 1$:*

If $f \in \ell^1$, then $\sum_{n=1}^{\infty} |\frac{1}{n} f(n)| \leq \sum_{n=1}^{\infty} |f(n)| = \|f\|_1 < \infty$. This shows that the function $T(f)$ belongs to ℓ^1, hence that T is well-defined (as a map from ℓ^1 to ℓ^1); moreover, as T is then clearly linear, this also shows that $\|T(f)\|_1 \leq \|f\|_1$ for all $f \in \ell^1$, hence that T is bounded with $\|T\| \leq 1$. If e_1 is the function in ℓ^1 given by $e_1(1) = 1$ and $e_1(n) = 0$ for all $n \geq 2$, we have that $\|e_1\|_1 = 1$, and $T(e_1) = e_1$, so $\|T(e_1)\| = 1$. Hence, $\|T\| \geq 1$. Altogether, we get that $\|T\| = 1$. Finally, it is clear that $\ker(T) = \{0\}$, hence T is injective.

b) *The range of T, i.e., $Z := T(\ell^1)$, is not closed in ℓ^1:*

We first observe that $\overline{Z} = \overline{T(\ell^1)} = \ell^1$. To show this, let $g \in \ell^1$, and define for each $k \in \mathbb{N}$ the function $g_k \in \ell^1$ defined by

$$g_k(n) = \begin{cases} n\, g(n) & \text{if } 1 \leq n \leq k, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$[T(g_k)](n) = \begin{cases} g(n) & \text{if } 1 \leq n \leq k, \\ 0 & \text{otherwise,} \end{cases}$$

so $\|g - T(g_k)\|_1 = \sum_{n=k+1}^{\infty} |g(n)| \to 0$ as $k \to \infty$. Thus, $g \in \overline{T(\ell^1)} = \overline{Z}$.
Next, we observe that $Z \neq \ell^1$: indeed, if $h \in \ell^1$ is given by $h(n) = 1/n^2$ for all $n \in \mathbb{N}$, then it is quite obvious that $h \notin T(\ell^1) = Z$. Thus, we get that $\overline{Z} = \ell^1 \neq Z$, so Z is not closed.

c) Since T is a bijection between ℓ^1 and Z, we may consider the inverse map $T^{-1} : Z \to \ell^1$.

The linear map T^{-1} is unbounded:

It is straighforward to check that $Z = \{g \in \ell^1 \mid \sum_{n=1}^{\infty} n\, g(n) < \infty\}$ and that $T^{-1}(g)$ is given by $[T^{-1}(g)](n) = n\, g(n)$ for all $n \in \mathbb{N}$. For each $k \in \mathbb{N}$, let e_k be given by

$$e_k(n) = \begin{cases} 1 & \text{if } n = k, \\ 0 & \text{otherwise.} \end{cases}$$

Then $e_k \in Z$, $\|e_k\|_1 = 1$ and $\|\, T^{-1}(e_k)\|_1 = k$ for every $k \in \mathbb{N}$. Thus,

$$\sup\{\|T^{-1}(g)\|_1 : g \in Z, \|g\|_1 \leq 1\} \;\geq\; k \;\text{ for every } k \in \mathbb{N},$$

which implies that T^{-1} is unbounded.

We note that this implies that Z is not closed: Assume (for contradiction) that Z was closed. Then Z would be a Banach space, and the open mapping theorem would then imply that T^{-1} is bounded, contradicting what we showed above.

Exercise 18 [= Exercise 2.3.8 in Pedersen's book]

Let X, Y be Banach spaces and $T \in \mathcal{B}(X,Y)$ be such that $T(X)$ is closed in Y. Then $T^*(Y^*) = \ker(T)^\perp$:

Let $\psi \in Y^*$. Then, for all $x \in \ker(T)$, we have

$$[T^*(\psi)](x) = (\psi \circ T)(x) = \psi(T(x)) = \psi(0) = 0.$$

Thus, $T^*(\psi) \in \ker(T)^\perp$. This shows that $T^*(Y^*) \subseteq \ker(T)^\perp$. To show the reverse inclusion, follow the sketch of the proof given in Pedersen's book.

Exercise 19 [= Exercise 2.4.4 in Pedersen's book]

Let X be a normed space and N be a subspace of X^*. Then N is separating for X if and only if N is weak*-dense in X^*:

Recall that $N^\perp = \{x \in X : \psi(x) = 0 \text{ for all } \psi \in N\}$. One easily sees that N is separating for X if and only if $N^\perp = \{0\}$. Setting $Z := \overline{N}^{\text{weak}^*} \subseteq X^*$, it therefore suffices to show that $N^\perp = \{0\}$ if and only if $Z = X^*$. Now, it is elementary to check that $Z^\perp = N^\perp$, and we have seen in a lecture that $Z = (N^\perp)^\perp$. Thus, if $N^\perp = \{0\}$, then $Z = \{0\}^\perp = X^*$. Conversely, if $Z = X^*$, then $N^\perp = Z^\perp = (X^*)^\perp = \{x \in X : \psi(x) = 0 \text{ for all } \psi \in X^*\} = \{0\}$ (since X^* is separating for X).

Exercise 21

Let H be a Hilbert space and M be a closed subspace of H. Consider the quotient space H/M with the quotient norm.
Then H/M is isometrically isomorphic with $M^\perp$ (the orthogonal complement of M in H):

Let $Q : H \to H$ denote the orthogonal projection of H onto $M^\perp$. Then Q is bounded, has range $M^\perp$ and $\ker(Q) = M$. So the map $\tilde{Q} : H/M \to M^\perp$, given by $\tilde{Q}(\xi + M) = Q(\xi)$ for all $\xi \in H$, is an isomorphism such that $\|\tilde{Q}\| = \|Q\|$. In fact, $\tilde{Q}$ is isometric: indeed, $P := I - Q$ is the orthogonal projection of H onto M, so for each $\xi \in H$ we have

$$\|\tilde{Q}(\xi + M)\| = \|Q(\xi)\| = \|(I - P)(\xi)\| = \|\xi - P(\xi)\| = \inf\{\|\xi - m\| : m \in M\} = \|\xi + M\|.$$

Note that H/M may be organized as a Hilbert space:
We can use $\tilde{Q}$ to transport the inner product of $M^\perp$ to X/M, by setting

$$\langle \xi + M, \eta + M \rangle := \langle \tilde{Q}(\xi + M), \tilde{Q}(\eta + M) \rangle = \langle Q(\xi), Q(\eta) \rangle$$

for all $\xi, \eta \in H$. It is then straightforward to check that H/M becomes a Hilbert space w.r.t. this inner product.

Exercise 22

Let X be a normed space. Let M be a closed subspace of X and let $Q : X \to X/M$ denote the quotient map. Then the adjoint operator $Q^* : (X/M)^* \to X^*$ is isometric:

Let $\psi \in (X/M)^*$. Since $\|Q^*\| = \|Q\| \leq 1$, we get that $\|Q^*(\psi)\| \leq \|Q^*\|\,\|\psi\| \leq \|\psi\|$.

To show the reverse inequality, we will use the fact that if X is a normed space and $\varphi \in X^*$, then

$$\|\varphi\| = \sup\{|\varphi(x)| : x \in X, \|x\| < 1\}.$$

(You should prove this fact if you have not seen it before.)

Let $y + M \in X/M$ satisfy that $\|y + M\| < 1$. By definition of the quotient norm, we can then pick $m \in M$ such that $\|y + m\| < 1$. Set $z := y + m \in X$. Then $\|z\| < 1$, and we get that

$$\begin{aligned} |\psi(y + M)| &= |\psi(z + M)| = |\psi(Q(z))| = |(Q^*(\psi))(z)| \\ &\leq \sup\left\{\big|[Q^*(\psi)](x)\big| : x \in X, \|x\| < 1\right\} \\ &= \|Q^*(\psi)\|. \end{aligned}$$

This gives that

$$\|\psi\| \ = \sup\left\{\big|\psi(y + M)\big| : y + M \in X/M,\ \|y + M\| < 1\right\} \leq \|Q^*(\psi)\|.$$

Altogether, we have shown that $\|Q^*(\psi)\| = \|\psi\|$, as desired.