MAT4450 - Spring 2024 - Solutions of exercises - Set 6

Exercise 23

Let X be a vector space (over $\mathbb{F}$).

a) Let A be a nonempty subset of X and set

$$\mathrm{co}(A) = \Big\{ \sum_{j=1}^{n} \lambda_j\, a_j \mid n \in \mathbb{N},\, \lambda_1, \ldots, \lambda_n \in [0,1],\, \sum_{j=1}^{n} \lambda_j = 1,\, a_1, \ldots, a_n \in A \Big\}.$$

(This says that $\mathrm{co}(A)$ consists of all possible convex combinations of vectors in A.)

Then co(A) is the least convex subset of X containing A:

We first remark that an easy induction argument shows that the following statement holds for every $n \in \mathbb{N}$: If C is a convex subset of X and $u_1, \ldots, u_n \in C$, then any convex combination of the u_j's belongs to C, that is, if $\lambda_1, \ldots, \lambda_n \in [0,1]$, and $\sum_{j=1}^{n} \lambda_j = 1$, then $\sum_{j=1}^{n} \lambda_j\, u_j \in C$.

Next, we note that $A \subseteq \mathrm{co}(A)$ (take $n = 1$). Moreover, $\mathrm{co}(A)$ is convex:

Indeed, if $x = \sum_{j=1}^{n} \lambda_j\, a_j$ and $y = \sum_{k=1}^{m} \mu_k\, b_k$ both lies in $\mathrm{co}(A)$ (with obvious assumptions on the a_j's, the b_k's, the λ_j's and the μ_k's), then for each $0 < t < 1$, we have

$$(1-t)x + ty = \sum_{j=1}^{n} (1-t)\lambda_j\, a_j + \sum_{k=1}^{m} t\mu_k\, b_k \in \mathrm{co}(A)$$

since

$$\sum_{j=1}^{n} (1-t)\lambda_j + \sum_{k=1}^{m} t\mu_k = 1 - t + t = 1.$$

Finally, assume that C is a convex subset of X containing A. Then $\mathrm{co}(A) \subseteq C$:
Indeed, assume that $x \in \mathrm{co}(A)$. Write $x = \sum_{j=1}^{n} \lambda_j\, a_j$ as a convex combination of $a_1, \ldots, a_n \in A$. Then $a_1, \ldots, a_n \in C$, so our first remark implies that $x \in C$.

This shows that $\mathrm{co}(A)$ is the least convex subset of X containing A, as desired.

b) Let C be a nonempty convex subset of X. Assume F is a face of C and K is a face of F.

Then K is a face of C:

Consider $x, y \in C$ and $0 < t < 1$ such that $(1-t)x + ty \in K$. Since $K \subseteq F$, we have $(1-t)x + ty \in F$. Since F is a face of C, this implies that $x, y \in F$. Since K is a face of F, we get that $x, y \in K$, as desired.

Exercise 24

Let Ω be a compact Hausdorff space and let $\{f_n\}$ be a sequence in $C(\Omega, \mathbb{F})$ such that $\sup_{n\in\mathbb{N}} \|f_n\|_\infty < \infty$ and $\{f_n\}$ converges to some $f \in C(\Omega, \mathbb{F})$ pointwise on Ω.

Then we have

$$f \in \overline{\mathrm{co}\big(\{f_n, n \in \mathbb{N}\}\big)}^{\|\cdot\|_\infty}.$$

To prove this assertion, we observe that, since $\mathrm{co}\big(\{f_n, n \in \mathbb{N}\}\big)$ is a convex subset of $C(\Omega, \mathbb{F})$, we know that

$$\overline{\mathrm{co}\big(\{f_n, n \in \mathbb{N}\}\big)}^{\|\cdot\|_\infty} = \overline{\mathrm{co}\big(\{f_n, n \in \mathbb{N}\}\big)}^{\mathrm{weak}}.$$

It therefore suffices to show that $f_n \to f$ as $n \to \infty$ in the weak topology. Let's assume that $\mathbb{F} = \mathbb{C}$. (The case where $\mathbb{F} = \mathbb{R}$ can be handled in a similar way.)

We recall that the dual space of $C(\Omega, \mathbb{C})$ may be identified with the space $M(\Omega)$ of all regular complex Borel measures on Ω. If $\mu \in M(\Omega) = C(\Omega, \mathbb{C})^*$ and $f \in C(\Omega, \mathbb{C})$, this identification means that we have $\mu(f) = \int_X f \, d\mu$. A natural idea to prove the desired statement is therefore to make use of Lebesgue's Dominated Convergence Theorem (LDCT).

Consider first $\mu \in M(\Omega)^+$. Set $s := \sup_{n\in\mathbb{N}} \|f_n\|_\infty$ (which is a finite number by assumption). Then we have that $|f_n(x)| \leq s$ for all $x \in \Omega$, so $|f_n| \leq s\, 1_\Omega$ on Ω for all $n \in \mathbb{N}$. Since μ is a finite measure, the function $s\, 1_\Omega$ is integrable w.r.t. to μ. Hence we may apply the LDCT and get that every f_n is integrable w.r.t. μ, and

$$\lim_{n\to\infty} \mu(f_n) = \lim_{n\to\infty} \int_\Omega f_n \, d\mu = \int_\Omega f \, d\mu = \mu(f).$$

Since any $\mu \in M(\Omega)$ is a linear combination of measures in $M(\Omega)^+$, we get that $\lim_{n\to\infty} \mu(f_n) = \mu(f)$ for all $\mu \in M(\Omega) = C(\Omega, \mathbb{C})^*$, i.e., $f_n \to f$ weakly, as desired.

Note: The assumptions in this exercise are not enough to guarantee that f_n converges uniformly to f as $n \to \infty$ (which would have solved this exercise in a rather trivial way). You can f.ex. consider $\Omega = [0, 1]$, and for each $n \in \mathbb{N}$ let $f_n : [0, 1] \to \mathbb{F}$ be a continuous function taking all its values in $[0, 1]$, which is zero outside $(1/(n + 1), 1/n)$ and takes the value 1 at the midpoint of $(1/(n + 1), 1/n)$. Then $\sup_{n\in\mathbb{N}} \|f_n\|_\infty = 1$ and f_n converges to 0 pointwise on Ω, but it does not converge to 0 uniformly on Ω.

Exercise 26

Consider the Banach space $X = (L^1(\mathbb{R}, \mathcal{B}_\mathbb{R}, \mu), \|\cdot\|_1)$ (over $\mathbb{F}$), where $\mathcal{B}_\mathbb{R}$ denotes the σ-algebra of all Borel subsets of $\mathbb{R}$ and μ denotes the Lebesgue measure on $\mathcal{B}_\mathbb{R}$.

Consider $B := \{f \in X : \|f\|_1 \leq 1\}$, which is clearly convex.

a) *The convex ball B has no extreme points*:

We follow the hint and assume (for contradiction) that there exists some $f \in \mathrm{ex}(B)$. We first note that $\|f\|_1 = 1$. Indeed, assume $t := \|f\|_1 \in [0, 1)$.

If $t = 0$, i.e., $f = 0$, then we can pick $g \in B \setminus \{0\}$, in which case we also have $-g \in B \setminus \{0\}$, and write $f = 0 = \frac{1}{2}g + \frac{1}{2}(-g)$, which shows that $f \notin \mathrm{ex}(B)$.

If $t \in (0, 1)$, then $f \neq 0$ and $f = t\,(\frac{1}{t}f) + (1 - t)\,0$, where $\frac{1}{t}f$ and 0 both lie in B, which shows that $f \notin \mathrm{ex}(B)$.

Thus, in both cases, we get a contradiction, i.e., we must have $t = 1$, as desired.

We now consider the continuous function $F(t) := \int_{(-\infty,t]} |f| \, d\mu,\ t \in \mathbb{R}$. Since

$$\lim_{t\to\infty} F(t) = \|f\|_1 = 1 \quad \text{and} \quad \lim_{t\to-\infty} F(t) = 0,$$

there exists (at least one) $t_0 \in \mathbb{R}$ such that $F(0) = 1/2$.

Set $f_1 := 2\, f\, \mathbf{1}_{(-\infty,t_0]}$ and $f_2 := 2\, f\, \mathbf{1}_{(t_0,\infty]}$. Then $f_1, f_2 \in B$ since

$$\|f_1\|_1 = 2\, F(t_0) = 2 \cdot \frac{1}{2} = 1, \quad \text{and}$$

$$\|f_2\|_1 = 2 \int_\mathbb{R} |f|\, \mathbf{1}_{(t_0,\infty]} \, d\mu = 2 \int_\mathbb{R} |f| \, d\mu - 2\, F(t_0) = 2 - 1 = 1.$$

Moreover, we have

$$\frac{1}{2} f_1 + \frac{1}{2} f_2 = f\, \mathbf{1}_{(-\infty,t_0]} + f\, \mathbf{1}_{(t_0,\infty]} = f.$$

As $f \in \text{ex}(B)$, this implies that $f = f_1 = f_2$, which is clearly possible only if $f = 0$. This contradicts that $\|f\|_1 = 1$. Hence, we must have that $\text{ex}(B) = \emptyset$.

b) *There is no topology on X making it a locally convex Hausdorff topological vector space such that B is compact*:

Assume τ was such a topology on X. The Krein-Milman theorem would then imply that B had some extreme points, which is not the case. Hence, there is no such topology on X.

This implies that $(L^1(\mathbb{R}, \mathcal{B}_\mathbb{R}, \mu), \|\cdot\|_1)$ can not be isomorphic to the dual space of $(L^\infty(\mathbb{R}, \mathcal{B}_\mathbb{R}, \mu), \|\cdot\|_\infty)$:

If $(L^1(\mathbb{R}, \mathcal{B}_\mathbb{R}, \mu), \|\cdot\|_1)$ was isomorphic to the dual space of $(L^\infty(\mathbb{R}, \mathcal{B}_\mathbb{R}, \mu), \|\cdot\|_\infty)$, then we could equip $X = L^1(\mathbb{R}, \mathcal{B}_\mathbb{R}, \mu)$ with the weak*-topology and B would then be compact by Alaoglu's theorem; this would contradict the statement above.

Exercise 27

Let (X, τ) be a locally convex Hausdorff topological vector space and K be a nonempty compact convex subset of X.
Let $\varphi \in (X, \tau)^*$ and set $m := \inf \operatorname{Re} \varphi(K)$, $M := \sup \operatorname{Re} \varphi(K)$, $s := \sup |\varphi(K)|$.

There exist $x, y, z \in \text{ex}(K)$ such that

$$\operatorname{Re} \varphi(x) = m, \ \operatorname{Re} \varphi(y) = M, \ |\varphi(z)| = s\,.$$

Set $F := \{x \in K : \operatorname{Re} \varphi(x) = m\}$. Then, as we have seen in a lecture, F is a compact face of K. Thus F is a compact convex subset of X, so the Krein-Milman theorem gives that $\text{ex}(F) \neq \emptyset$. Letting $x \in \text{ex}(F)$, we get that $x \in \text{ex}(K)$ and $\operatorname{Re} \varphi(x) = m$.

Next, set $\psi := -\varphi$. Then $\operatorname{Re} \psi = -\operatorname{Re} \varphi$ and

$$\inf \operatorname{Re} \psi(K) = -\sup \operatorname{Re} \varphi(K) = -M.$$

Using what we have shown above (with ψ instead of φ), we get that there exists $y \in \text{ex}(K)$ such that $\operatorname{Re} \psi(y) = -M$, i.e., $\operatorname{Re} \varphi(y) = M$.

Finally, to show the existence of z, we consider the function $|\varphi|$ on X. Since K is compact, there exists some $x_0 \in K$ such that

$$|\varphi(x_0)| = s \ (= \sup |\varphi(K)|).$$

Pick then $t \in \mathbb{R}$ such that $|\varphi(x_0)| = e^{it} \varphi(x_0)$, and define $\varphi' \in (X, \tau)^*$ by $\varphi' := e^{it} \varphi$.
Note that $\varphi'(x_0) = |\varphi(x_0)| = s \in \mathbb{R}$, so $\operatorname{Re} \varphi'(x_0) = s$. Moreover, for every $v \in K$, we have

$$\operatorname{Re} \varphi'(v) \ \leq \ |\varphi'(v)| = |\varphi(v)| \ \leq \ s.$$

Thus, $\sup \operatorname{Re} \varphi'(K) = s$. Using the second assertion we have shown (with φ' instead of φ), we get that there exists $z \in \text{ex}(K)$ such that $\operatorname{Re} \varphi'(z) = s$. This gives that

$$s = \operatorname{Re} \varphi'(z) \leq |\varphi'(z)| = |\varphi(z)| \leq s,$$

hence that $|\varphi(z)| = s$, as desired.