MAT4450 - Spring 2024 - Solutions of exercises - Set 7

Exercise 28

Let K be a nonempty compact subset of $\mathbb{R}^2$, and consider the set $\mathcal{P}$ of all real polynomials in two commuting variables as a subset of $C(K, \mathbb{R})$.

Then $\mathcal{P}$ is dense in $C(K, \mathbb{R})$ (w.r.t. $\|\cdot\|_\infty$):

One readily checks that $\mathcal{P}$ is a subalgebra of $C(K, \mathbb{R})$. Moreover, $\mathcal{P}$ separates the points of K: Indeed, assume that $(x_1, y_1), (x_2, y_2) \in K$, $(x_1, y_1) \neq (x_2, y_2)$. Let then $p \in \mathcal{P}$ be given by $p(x, y) = (x - x_1)^2 + (y - y_1)^2$ for all $(x, y) \in K$. Then $p(x_1, y_1) = 0$, while $p(x_2, y_2) > 0$. Thus, $p(x_1, y_1) \neq p(x_2, y_2)$.

Since we also have that $1_K \in \mathcal{P}$, the (real) Stone-Weierstrass theorem gives the desired conclusion.

Exercise 29

Let Ω be a compact Hausdorff space, and let $\mathcal{A}$ be a subalgebra of $C(\Omega, \mathbb{R})$.

a) Let $f \in \mathcal{A}$ and $n \in \mathbb{N}$. *Then $|f|^{1/n} \in \overline{\mathcal{A}}^{\|\cdot\|_\infty}$:*

Since $f(\Omega)$ is a compact subset of $\mathbb{R}$, we may choose $m > 0$ such that $f(\Omega) \subseteq [-m, m]$. Now let $g : [-m, m] \to [0, \infty)$ be the continuous function given by $g(x) = |x|^{1/n}$. Since $g(0) = 0$, using Weierstrass' theorem, we may find a sequence $\{p_k\}$ of real polynomials such that $p_k(0) = 0$ for all k and $p_k \to g$ uniformly on $[-m, m]$. It follows that $p_k \circ f \in \mathcal{A}$ and

$$\| \, |f|^{1/n} - p_k \circ f\|_\infty = \|(g - p_k) \circ f\|_\infty \leq \sup\{|(g - p_k)(t)| : t \in [-m, m]\} \to 0$$

as $k \to \infty$. Hence, $|f|^{1/n} \in \overline{\mathcal{A}}^{\|\cdot\|_\infty}$.

b) Assume that $\mathcal{A}$ separates the points of Ω and that there exists some $g \in \mathcal{A}$ such that $g(x) \neq 0$ for all $x \in \Omega$. *Then $\overline{\mathcal{A}}^{\|\cdot\|_\infty} = C(\Omega, \mathbb{R})$:*

We first show that $1_\Omega \in \overline{\mathcal{A}}^{\|\cdot\|_\infty}$. Replacing g with $\frac{1}{\|g\|_\infty}$ if necessary, we can assume that $|g| \leq 1$ on Ω. Since $0 \notin |g|(\Omega)$, we have that $\delta := \inf |g|(\Omega) > 0$. Thus we get that

$$\|1_\Omega - |g|^{1/n}\|_\infty = \sup_{\omega \in \Omega} 1 - |g(\omega)|^{1/n} \leq 1 - \delta^{1/n} \ \to 0 \quad \text{as} \quad n \to \infty,$$

which shows that $1_\Omega \in \overline{\mathcal{A}}^{\|\cdot\|_\infty}$.

It follows now that $\mathcal{B} := \overline{\mathcal{A}}^{\|\cdot\|_\infty}$ is a subalgebra of $C(\Omega, \mathbb{R})$ satisfying all the conditions in the real version of the Stone-Weierstrass theorem. Thus we get that $C(\Omega, \mathbb{R}) = \overline{\mathcal{B}}^{\|\cdot\|_\infty} = \overline{\mathcal{A}}^{\|\cdot\|_\infty}$, as desired.

Exercise 30.

Let H denote a nontrivial (complex) Hilbert space. Recall that $\mathcal{K}(H)$ denotes the compact linear operators on H, and that it is a Banach algebra (w.r.t. operator norm).

Then $\mathcal{K}(H)$ is unital if and only if H is finite-dimensional:

If H is finite-dimensional, then $\mathcal{K}(H) = \mathcal{B}(H)$ is unital with unit I_H (the identity operator on H). Conversely, assume $\mathcal{K}(H)$ is unital with unit I. Let $\xi \in H$ and let P denote the finite-rank projection given by $P(\eta) = \langle \eta, \xi \rangle \xi$ for all $\eta \in H$. Since $P \in \mathcal{K}(H)$, we have $PI = IP = P$. This implies that $I(\xi) = I(P(\xi)) = (IP)(\xi) = P(\xi) = \xi$. Hence, $I = I_H$, so I_H is compact, which we know happens (if and) only if H is finite-dimensional.

Exercise 31.

a) Consider

$$C_0(\mathbb{R},\mathbb{C}) := \{f : \mathbb{R} \to \mathbb{C} \mid f \text{ is continuous and } \lim_{t\to\infty} f(t) = \lim_{t\to-\infty} f(t) = 0\}$$

(w.r.t. $\|\cdot\|_\infty$).

• *$C_0(\mathbb{R},\mathbb{C})$ is a norm-closed subalgebra of $C_b(\mathbb{R},\mathbb{C})$*: This is quite elementary so we skip the proof.

Since $C_b(\mathbb{R},\mathbb{C})$ is a commutative Banach algebra, we get that $C_0(\mathbb{R},\mathbb{C})$ is a commutative Banach algebra.

• *$C_0(\mathbb{R},\mathbb{C})$ is non-unital*:

Assume (for contradiction) that $C_0(\mathbb{R},\mathbb{C})$ is unital with unit I. Let $x \in \mathbb{R}$. Define then $f : \mathbb{R} \to \mathbb{C}$ by

$$f(t) := \begin{cases} t-(x-1) & \text{if } x-1 \le t \le x, \\ (x+1)-t & \text{if } x \le t \le x+1, \\ 0 \quad \text{otherwise.} \end{cases}$$

for $t \in \mathbb{R}$. Then $f \in C_0(\mathbb{R},\mathbb{C})$ and $f(x) = 1$. So $I(x) = I(x)f(x) = (If)(x) = f(x) = 1$. Since this holds for every $x \in \mathbb{R}$, we get that $I = 1_\mathbb{R}$, which is impossible since $1_\mathbb{R} \notin C_0(\mathbb{R},\mathbb{C})$.

b) More generally, let Ω denote a locally compact Hausdorff space. If $f : \Omega \to \mathbb{C}$, then say that f vanishes at infinity if for every $\varepsilon > 0$ the set $\{x \in \Omega : |f(x)| \ge \varepsilon\}$ is compact in Ω. Set

$$C_0(\Omega,\mathbb{C}) := \{f : \Omega \to \mathbb{C} \mid f \text{ is continuous and vanishes at infinity}\}.$$

• *$C_0(\Omega,\mathbb{C})$ is a norm-closed subalgebra of $C_b(\Omega,\mathbb{C})$*:

It is quite tedious to prove this directly. We sketch a more elegant way. If Ω is compact, then $C_0(\Omega,\mathbb{C}) = C(\Omega,\mathbb{C}) = C_b(\Omega,\mathbb{C})$. So we can assume that Ω is not compact and let $\widetilde{\Omega} = \Omega \cup \{\infty\}$ denote the one-point compactification of Ω. Set $\mathcal{J} := \{g \in C(\widetilde{\Omega},\mathbb{C}) : g(\infty) = 0\}$, which is a closed ideal of $C(\widetilde{\Omega},\mathbb{C})$. Moreover, let $\phi : C(\widetilde{\Omega},\mathbb{C}) \to C_b(\Omega,\mathbb{C})$ denote the algebra-homomorphism given by $\phi(g) = g_{|\Omega}$ (= the restriction of g to Ω) for each $g \in C(\widetilde{\Omega},\mathbb{C})$. It is not difficult to check that $\|\phi(g)\|_\infty = \|g\|_\infty$ for every $g \in \mathcal{J}$, and that $\phi(\mathcal{J}) = C_0(\Omega,\mathbb{C})$. It follows that $C_0(\Omega,\mathbb{C})$ is a subalgebra of $C_b(\Omega,\mathbb{C})$, which is complete (since $\mathcal{J}$ is complete), hence norm-closed.

Since $C_b(\Omega,\mathbb{C})$ is a commutative Banach algebra, we get that $C_0(\Omega,\mathbb{C})$ is a commutative Banach algebra.

• *$C_0(\Omega,\mathbb{C})$ is unital if and only if Ω is compact*:

If Ω is compact, then $C_0(\Omega,\mathbb{C}) = C(\Omega,\mathbb{C})$ is unital with unit 1_Ω. Conversely, assume $C_0(\Omega,\mathbb{C})$ is unital with unit I. Let $\omega \in \Omega$. We may then pick $f \in C_0(\Omega,\mathbb{C})$ such that $f(\omega) = 1$ (this follows for example from Proposition 1.7.5 in Pedersen's book). Then
$I(\omega) = I(\omega)f(\omega) = (If)(\omega) = f(\omega) = 1$. Thus, $I = 1_\Omega$. So 1_Ω vanishes at infinity, which happens (if and) only if Ω is compact.