MAT4450 - Spring 2024 - Solutions of exercises - Set 8

Exercise 32

Consider the commutative Banach algebra $\mathcal{A} = C(\Omega, \mathbb{F})$, *where* Ω *is a compact Hausdorff space. Let* $\omega_0 \in \Omega$ *and define* $\varphi : \mathcal{A} \to \mathbb{F}$ *by* $\varphi(f) = f(\omega_0)$ *for* $f \in \mathcal{A}$.

a) *Then* φ *is a continuous algebra-homomorphism from* $\mathcal{A}$ *into* $\mathbb{F}$ *(considered as a Banach algebra) satisfying* $\|\varphi\| = 1$:

It is clear that φ is linear. It is also multiplicative. Indeed, let $f, g \in \mathcal{A}$. Then

$$\varphi(fg) = (fg)(\omega_0) = f(\omega_0)\, g(\omega_0) = \varphi(f)\varphi(g).$$

Thus, φ is an algebra-homomorphism. Moreover, φ is continuous (= bounded), with $\|\varphi\| \leq 1$, since $|\varphi(f)] = |f(\omega_0)| \leq \|f\|_\infty$ for all $f \in \mathcal{A}$. As $\|1_\Omega\|_\infty = 1$ and $|\varphi(1_\Omega)| = 1_\Omega(\omega_0) = 1$, we get that $\|\varphi\| = 1$.

b) *Consider the closed ideal of* $\mathcal{A}$ *given by* $\mathcal{J} = \ker(\varphi)$. *Then the Banach algebra* $\mathcal{A}/\mathcal{J}$ *is isometrically isomorphic to* $\mathbb{F}$:

Since the range of φ is $\mathbb{F}$, we get that the map $\widetilde{\varphi} : \mathcal{A}/\mathcal{J} \to \mathbb{F}$ given by $\widetilde{\varphi}(f + \mathcal{J}) = \varphi(f)$ for each $f \in \mathcal{A}$ is a bounded algebra-isomorphism (with $\|\widetilde{\varphi}\| = \|\varphi\| = 1$).

Let $f \in \mathcal{A}$. To show that $\widetilde{\varphi}$ is isometric, we have to show that $|\widetilde{\varphi}(f + \mathcal{J})| = \|f + \mathcal{J}\|$. Since $|\widetilde{\varphi}(f + \mathcal{J})| = |\varphi(f)| = |f(\omega_0)|$ and $\|f + \mathcal{J}\| = \inf\{\|f - g\|_\infty : g \in \mathcal{J}\}$, we have to show that

$$|f(\omega_0)| = \inf\{\|f - g\|_\infty : g \in \mathcal{J}\}.$$

Since $\mathcal{J} = \{g \in \mathcal{A} \mid g(\omega_0) = 0\}$, we have that

$$|f(\omega_0)| = |(f - g)(\omega_0)| \leq \|f - g\|_\infty \quad \text{for all } g \in \mathcal{J},$$

Hence, $|f(\omega_0)|$ is a lower bound of $\{\|f - g\|_\infty : g \in \mathcal{J}\}$.

Let now $\varepsilon > 0$. Since f is continuous at ω_0, we can find an open neighborhood U of ω_0 such that $|f(\omega) - f(\omega_0)| < \varepsilon$ for all $\omega \in U$. This gives that

$$\Big| |f(\omega)| - |f(\omega_0)| \Big| \leq |f(\omega) - f(\omega_0)| < \varepsilon \quad \text{for all } \omega \in U,$$

hence that $|f(\omega)| < |f(\omega_0)| + \varepsilon$ for all $\omega \in U$.

Set $E := \Omega \setminus U$. Then E and $\{\omega_0\}$ are disjoint closed subsets of Ω, so, by Urysohn's lemma, we can find $h \in \mathcal{A}$ such that $0 \leq h \leq 1$, $h = 1$ on E, and $h(\omega_0) = 0$.

Set $k := fh$. Then $k \in \mathcal{J}$, $f - k = 0$ on E, and

$$|(f - k)(\omega)| = |f(\omega)|(1 - h(\omega)) \leq |f(\omega)| < |f(\omega_0)| + \varepsilon \quad \text{for all } \omega \in U.$$

Thus we get that

$$\|f - k\|_\infty \leq |f(\omega_0)| + \varepsilon.$$

Since $k \in \mathcal{J}$, it follows that $|f(\omega_0)|$ is the greatest lower bound of $\{\|f - g\|_\infty : g \in \mathcal{J}\}$, as desired.

Exercise 33

Let $\mathcal{A}$ be a non-unital Banach algebra over $\mathbb{F}$. Set $\widetilde{\mathcal{A}} = \{(a, \alpha) \mid a \in \mathcal{A}, \alpha \in \mathbb{F}\}$ and define addition, multiplication by scalars and product by

$$\begin{aligned}(a,\alpha) + (b,\beta) &= (a+b, \alpha+\beta),\\ \lambda\,(a,\alpha) &= (\lambda a, \lambda\alpha),\\ (a,\alpha)(a,\beta) &= (ab + \alpha b + \beta a, \alpha\beta)\end{aligned}$$

for $(a,\alpha),(b,\beta) \in \widetilde{\mathcal{A}}$ and $\lambda \in \mathbb{F}$. As you can check for yourself, $\widetilde{\mathcal{A}}$ becomes a unital algebra with unit $e := (0,1)$, and that the map $(a,\alpha) \mapsto \alpha$ is an algebra-homomorphism from $\widetilde{\mathcal{A}}$ into $\mathbb{F}$, with kernel equal to $\mathcal{A}_0 := \{(a,0) \mid a \in \mathcal{A}\}$.
We identify $\mathcal{A}$ with the ideal $\mathcal{A}_0$ and write $a + \alpha\, e$ instead of (a,α).

For $a + \alpha\tilde{e} \in \widetilde{\mathcal{A}}$, set $\|a + \alpha e\| := \|a\| + |\alpha|$.

Then $\widetilde{\mathcal{A}}$ is a unital Banach algebra w.r.t. $\|\cdot\|$, which contains $\mathcal{A}$ as a closed ideal:

Let $a + \alpha e, b + \beta e \in \widetilde{\mathcal{A}}$ and $\lambda \in \mathbb{C}$. Then $\|a + \alpha e\| = \|a\| + |\alpha| \geq 0$,

$$\|\lambda(a + \alpha e)\| = \|\lambda a + \lambda\alpha e\| = \|\lambda a\| + |\lambda\alpha| = |\lambda|(\|a\| + |\alpha|) = |\lambda|\|a + \alpha e\|,$$

$$\|a + \alpha e\| = 0 \Leftrightarrow \|a\| = 0 \text{ and } |\alpha| = 0 \Leftrightarrow a = 0 \text{ and } \alpha = 0 \Leftrightarrow a + \alpha e = 0,$$

$$\begin{aligned}\|(a+\alpha e) + (b + \beta e)\| &= \|a + b + (\alpha+\beta)e\| = \|a+b\| + |\alpha+\beta|\\ &\leq \|a\| + \|b\| + |\alpha| + |\beta| = (\|a\| + |\alpha|) + (\|b\| + |\beta|)\\ &= \|a + \alpha e\| + \|b + \beta e\|,\end{aligned}$$

$$\begin{aligned}\|(a+\alpha e)(b+\beta e)\| &= \|ab + \alpha b + \beta a + \alpha\beta e\|\\ &= \|ab + \alpha b + \beta a\| \ + |\alpha\beta|\\ &\leq \|ab\| + \|\alpha b\| + \|\beta a\| + |\alpha||\beta|\\ &\leq \|a\|\|b\| + |\alpha|\,\|b\| + |\beta|\|a\| + |\alpha||\beta|\\ &= (\|a\| + |\alpha|)(\|b\| + |\beta|)\\ &= \|a + \alpha e\|\|b + \beta e\|.\end{aligned}$$

This shows that $\|\cdot\|$ is a norm on $\widetilde{\mathcal{A}}$ which is submultiplicative, i.e., $\widetilde{\mathcal{A}}$ is a normed algebra. Further, $(\mathcal{A}, \|\cdot\|)$ is complete. Indeed, let $\{a_n + \alpha_n e\}$ be a Cauchy sequence in $\widetilde{\mathcal{A}}$. As

$$\|\ (a_n + \alpha_n e) - (a_m + \alpha_m e)\| = \|a_n - a_m\| \ + |\alpha_n - \alpha_m|$$

for all $n, m \in \mathbb{N}$, we see that $\{a_n\}$ and $\{\alpha_n\}$ are both Cauchy sequences, in $\mathcal{A}$ and in $\mathbb{F}$, respectively. So there exist $a \in \mathcal{A}$ and $\alpha \in \mathbb{F}$ such that $a_n \to a$ and $\alpha_n \to \alpha$ as $n \to \infty$, and it follows then readily that $a_n + \alpha_n e \to a + \alpha e$ as $n \to \infty$. This shows that $\widetilde{\mathcal{A}}$ is complete as a normed space, hence that $\widetilde{\mathcal{A}}$ is a Banach algebra.

As the map $A + \alpha e \mapsto \alpha$ clearly gives a bounded algebra-homomorphism π from $\widetilde{\mathcal{A}}$ into $\mathbb{F}$, having $\mathcal{A}$ as its kernel, we get that $\mathcal{A}$ is closed ideal of $\widetilde{\mathcal{A}}$.

Moreover, the quotient $\widetilde{\mathcal{A}}/\mathcal{A}$ is isometrically isomorphic to $\mathbb{F}$:

Let $\pi : \widetilde{\mathcal{A}} \to \mathbb{F}$ be the algebra-homomorphism given by $\pi(a + \alpha e) := \alpha$. Since the range of π is $\mathbb{F}$, we get that the map $\widetilde{\pi} : \widetilde{\mathcal{A}}/\mathcal{A} \to \mathbb{F}$ given by

$$\widetilde{\pi}((a + \alpha e) + \mathcal{A})) := \pi(a + \alpha e) = \alpha$$

is an algebra-isomorphism. For each $a + \alpha e \in \widetilde{\mathcal{A}}$, we have that

$$\|(a+\alpha e)+\mathcal{A}\| = \inf\{\|a+\alpha e-b\| : b \in \mathcal{A}\} = \inf\{\|a-b\|+|\alpha| : b \in \mathcal{A}\} = |\alpha| = \|\widetilde{\pi}((a+\alpha e)+\mathcal{A}))\|,$$

so $\widetilde{\pi}$ is isometric.

From now on, a unital Banach algebra $\mathcal{A}$ will always mean a (complex) Banach algebra having a unit $1_{\mathcal{A}}$ which satisfies that $\|1_{\mathcal{A}}\| = 1$.

Exercise 34

Let $\mathcal{A}$ be a unital Banach algebra.

a) Let $a \in \mathrm{GL}(\mathcal{A})$ and note that $0 \notin \mathrm{sp}_{\mathcal{A}}(a)$. *We have that* $\mathrm{sp}_{\mathcal{A}}(a^{-1}) = \{\, \lambda^{-1} \mid \lambda \in \mathrm{sp}_{\mathcal{A}}(a)\,\}$:

Let $\lambda \in \mathbb{C} \setminus \{0\}$. Since $\lambda^{-1}I - a^{-1} = -\lambda^{-1}a^{-1}(\lambda I - a)$, and $\lambda^{-1}a^{-1} \in \mathrm{GL}(\mathcal{A})$, we get that

$$\lambda \in \mathrm{sp}_{\mathcal{A}}(a) \Leftrightarrow \lambda^{-1} \in \mathrm{sp}_{\mathcal{A}}(a^{-1})$$

This implies that

$$\{\, \lambda^{-1} \mid \lambda \in \mathrm{sp}_{\mathcal{A}}(a)\,\} \subseteq \mathrm{sp}_{\mathcal{A}}(a^{-1}).$$

On the other hand, let $\mu \in \mathrm{sp}_{\mathcal{A}}(a^{-1})$. Then $\mu \neq 0$, and with $\lambda := \mu^{-1}$, we have that $\lambda^{-1} = \mu \in \mathrm{sp}_{\mathcal{A}}(a^{-1})$. As pointed out above, this implies that $\lambda \in \mathrm{sp}_{\mathcal{A}}(a)$. Thus, $\mu = \lambda^{-1}$ for some $\lambda \in \mathrm{sp}_{\mathcal{A}}(a)$, which shows that the reverse inclusion also holds.

b) Let $\mathcal{B}$ be unital Banach algebra and assume that $\phi : \mathcal{A} \to \mathcal{B}$ is an algebra-isomorphism such that $\phi(1_{\mathcal{A}}) = \phi(1_{\mathcal{B}})$.

Then we have that $\mathrm{sp}_{\mathcal{A}}(a) = \mathrm{sp}_{\mathcal{B}}(\phi(a))$ *for all* $a \in \mathcal{A}$:

Let $a \in \mathcal{A}$. Since $\phi(1_{\mathcal{A}}) = 1_{\mathcal{B}}$, it follows readily that if $a' \in \mathcal{A}$, then $a' \in \mathrm{GL}(\mathcal{A}) \Leftrightarrow \phi(a') \in \mathrm{GL}(\mathcal{B})$. So for $\lambda \in \mathbb{C}$ we get that

$$\lambda \in \mathrm{sp}_{\mathcal{A}}(a) \Leftrightarrow \lambda 1_{\mathcal{A}} - a \notin \mathrm{GL}(\mathcal{A}) \Leftrightarrow \phi(\lambda 1_{\mathcal{A}} - a) \notin \mathrm{GL}(\mathcal{B}) \Leftrightarrow \lambda 1_{\mathcal{B}} - \phi(a) \notin \mathrm{GL}(\mathcal{B}) \Leftrightarrow \lambda \in \mathrm{sp}_{\mathcal{B}}(\phi(a)),$$

which shows the assertion.

Note that we don't need to assume that $\phi(1_{\mathcal{A}}) = 1_{\mathcal{B}}$, because thi is automatically satisfied for any algebra-isomorphism $\phi : \mathcal{A} \to \mathcal{B}$. Indeed, $\phi(1_{\mathcal{A}})$ is then a unit for $\phi(\mathcal{A}) = \mathcal{B}$, so it must be equal to $1_{\mathcal{B}}$, by uniqueness of the unit in $\mathcal{B}$.

Exercise 34

Consider the complex Hilbert space $H = L^2([0,1], \mathcal{B}_{[0,1]}, \mu)$, where μ denotes the Lebesgue measure on the Borel σ-algebraen $\mathcal{B}_{[0,1]}$.

Set $\mathcal{A} := \mathcal{B}(H)$, and let $M \in \mathcal{A}$ denote the multiplication operator given by

$$[M(g)](t) = tg(t) \qquad \text{for all } g \in H \text{ and } t \in [0,1].$$

Then $\mathrm{sp}_{\mathcal{A}}(M) = [0,1]$:

We first show that $\mathrm{sp}_{\mathcal{A}}(M) \subseteq [0,1]$:

Let $\lambda \in \mathbb{C} \setminus [0,1]$. Then the function $g : [0,1] \to \mathbb{C}$ defined by $g(t) = (\lambda - t)^{-1}$ is continuous, so the multiplication operator $G : H \to H$ associated to g is bounded. As we clearly have that $(\lambda I - M)G = G(\lambda I - M) = I_H$, we get that $\lambda \notin \mathrm{sp}_{\mathcal{A}}(M)$. This shows that the inclusion above holds.

To show the reverse inclusion, let $\lambda \in [0,1]$. We will show that there exists a sequence $\{\xi_n\}$ of *unit* vectors in H such that

$$\|(\lambda I_H - M)\xi_n\|_2 \to 0 \quad \text{as } n \to \infty.$$

This will imply that $\lambda \in \mathrm{sp}_{\mathcal{A}}(M)$, because, otherwise, we would get that

$$1 = \|\xi_n\|_2 = \|(\lambda I_H - M)^{-1}(\lambda I_H - M)\xi_n\|_2 \le \|(\lambda I_H - M)^{-1}\| \, \|(\lambda I_H - M)\xi_n\|_2 \longrightarrow 0 \quad \text{as } n \to \infty,$$

giving a contradiction.

Assume first that $0 < \lambda < 1$ and choose $N \in \mathbb{N}$ such that $[\lambda, \lambda + 1/N] \subseteq [0,1]$. For each $n \ge N$, define $\xi_n : [0,1] \to \mathbb{C}$ by $\xi_n = n^{1/2}\, \mathbf{1}_{[\lambda,\, \lambda+1/n]}$. Then for each $n \ge N$ we have $\|\xi_n\|_2 = 1$ and

$$\|(\lambda I_H - M)\xi_n\|_2^2 = \frac{1}{n}\int_{\lambda}^{\lambda+1/n} (\lambda - t)^2\, dt = n\Big[\frac{1}{3}(t-\lambda)^3\Big]_{t=\lambda}^{t=\lambda+1/n} = \frac{1}{3n^2} \longrightarrow 0 \quad \text{as } n \to \infty.$$

We can clearly proceed similarly when $\lambda = 1$ by considering $\xi_n = n^{1/2}\, \mathbf{1}_{[\lambda-1/n,\, \lambda]}$. Thus, in both cases, we can conclude that $\lambda \in \mathrm{sp}_{\mathcal{A}}(M)$.

This shows that $[0,1] \subseteq \mathrm{sp}_{\mathcal{A}}(M)$. Altogether, the desired equality follows.

The operator M has no eigenvalues:

Assume $\lambda \in \mathbb{C}$ satisfies that $M\xi = \lambda\xi$ for some $\xi \in H$. Then we have that $t\xi(t) = \lambda\xi(t)$ for almost all $t \in [0,1]$, i.e., for all t belonging to some Borel set $A \subseteq [0,1]$ satisfying that $\mu(A) = 1$. This implies that $\xi(t) = 0$ for every $t \in A \setminus \{\lambda\}$. Since $\mu(A \setminus \{\lambda\}) = 1$, this means that $\xi = 0$ (μ-almost everywhere). This shows that no $\lambda \in \mathbb{C}$ can be an eigenvalue of M.

Exercise 35

Let S be a nonempty set and consider the unital Banach algebra $\mathcal{A} = \ell^\infty(S)$. Let $f \in \mathcal{A}$.

Then $\mathrm{sp}_{\mathcal{A}}(f) = \overline{f(S)}$:

We show below that this assertion holds when Ω is a topological space and $\mathcal{A} = C_b(\Omega)$ (= all bounded continuous complex functions on Ω) is equipped with the uniform norm $\|\cdot\|_\infty$.
(If S is a set, we can consider it as a topological space w.r.t. the discrete topology, and we then have $\ell^\infty(S) = C_b(S)$).

If $\lambda = f(\omega)$ for some $\omega \in \Omega$, then $(\lambda 1_\Omega - f)(\omega) = 0$, so $\lambda 1_\Omega - f \notin \mathrm{GL}(\mathcal{A})$, i.e., $\lambda \in \mathrm{sp}_{\mathcal{A}}(f)$. This shows that $f(\Omega) \subseteq \mathrm{sp}_{\mathcal{A}}(f)$. Since $\mathrm{sp}_{\mathcal{A}}(f)$ is closed in $\mathbb{C}$, we get that $\overline{f(\Omega)} \subseteq \mathrm{sp}_{\mathcal{A}}(f)$.

On the other hand, assume $\lambda \in \mathbb{C} \setminus \overline{f(\Omega)}$. Since $K := \overline{f(\Omega)}$ is closed, we have that $d := \inf |\lambda - z| : z \in K > 0$.

Define $g : \Omega \to \mathbb{C}$ by $g(\omega) = (\lambda - f(\omega))^{-1}$. Then

$$\|g\|_\infty = \sup\{|\lambda - f(\omega)|^{-1} : \omega \in \Omega\} \le \sup\{|\lambda - z|^{-1} : z \in K\} = d^{-1} < \infty,$$

so $g \in \mathcal{A}$. Moreover, it is then clear that g is the inverse of $\lambda 1_\Omega - f$ in $\mathcal{A}$, i.e., $\lambda \notin \mathrm{sp}_{\mathcal{A}}(f)$. This shows that $\mathrm{sp}_{\mathcal{A}}(f) \subseteq \overline{f(\Omega)}$.

Altogether, we get that $\mathrm{sp}_{\mathcal{A}}(f) = \overline{f(\Omega)}$, as desired.