MAT4450 - Spring 2024 - Solutions of exercises - Set 9

Exercise 37 [= Exercise 4.1.3 in Pedersen's book]

Let a and b be elements in a complex, unital Banach algebra $\mathcal{A}$. Then we have that

$$\mathrm{sp}(ab) \setminus \{0\} = \mathrm{sp}(ba) \setminus \{0\}.$$

Let $\lambda \in \mathrm{sp}(ab) \setminus \{0\}$. Set $c := \lambda^{-1}(1_{\mathcal{A}} + b(\lambda 1_{\mathcal{A}} - ab)^{-1}a) \in \mathcal{A}$. Then

$$\begin{aligned}(\lambda 1_{\mathcal{A}} - ba)c &= (\lambda 1_{\mathcal{A}} - ba)\lambda^{-1}(1_{\mathcal{A}} + b(\lambda 1_{\mathcal{A}} - ab)^{-1}a) \\ &= 1_{\mathcal{A}} + b(\lambda 1_{\mathcal{A}} - ab)^{-1}a - \lambda^{-1}ba - \lambda^{-1}bab(\lambda 1_{\mathcal{A}} - ab)^{-1}a\end{aligned}$$

Now

$$\begin{aligned}ab(\lambda 1_{\mathcal{A}} - ab)^{-1} &= \big(\lambda 1_{\mathcal{A}} - (\lambda 1_{\mathcal{A}} - ab)\big)(\lambda 1_{\mathcal{A}} - ab)^{-1} \\ &= \lambda(\lambda 1_{\mathcal{A}} - ab)^{-1} - 1_{\mathcal{A}}\end{aligned}$$

Hence

$$\begin{aligned}(\lambda 1_{\mathcal{A}} - ba)c &= 1_{\mathcal{A}} + b(\lambda 1_{\mathcal{A}} - ab)^{-1}a - \lambda^{-1}ba - \lambda^{-1}b\big(\lambda(\lambda 1_{\mathcal{A}} - ab)^{-1} - 1_{\mathcal{A}}\big)a \\ &= 1_{\mathcal{A}} + b(\lambda 1_{\mathcal{A}} - ab)^{-1}a - \lambda^{-1}ba - b(\lambda 1_{\mathcal{A}} - ab)^{-1}a + \lambda^{-1}ba \\ &= 1_{\mathcal{A}}\end{aligned}$$

Similarly, we get that $c(\lambda 1_{\mathcal{A}} - ba) = 1_{\mathcal{A}}$. Thus, $\lambda 1_{\mathcal{A}} - ba \in \mathrm{GL}(\mathcal{A})$, i.e., $\lambda \in \mathrm{sp}(ba) \setminus \{0\}$.

This shows that $\mathrm{sp}(ab) \setminus \{0\} \subseteq \mathrm{sp}(ba) \setminus \{0\}$. By symmetry, the reverse inclusion also holds, so the desired equality follows.

Exercise 38

Let X be a complex Banach space and consider the Banach algebra $\mathcal{A} := \mathcal{B}(X)$. Let $T \in \mathcal{A}$. The adjoint operator T^* belongs then to the Banach algebra $\mathcal{B} := \mathcal{B}(X^*)$.

We have that $sp_{\mathcal{B}}(T^*) = sp_{\mathcal{A}}(T)$:

We will first show that

$$T \in \mathrm{GL}(\mathcal{B}(X)) \;\Leftrightarrow\; T^* \in \mathrm{GL}(\mathcal{B}(X^*)). \tag{1}$$

Assume $T \in \mathrm{GL}(\mathcal{B}(X))$ and set $S = T^{-1} \in \mathcal{B}(X)$. Then we get that

$$T^*S^* = (ST)^* = (I_X)^* = I_{X^*},$$

Similarly, $S^*T^* = I_{X^*}$. Thus, $T^* \in \mathrm{GL}(\mathcal{B}(X^*)$. This shows the forward implication.

Conversely, assume $T^* \in \mathrm{GL}(\mathcal{B}(X^*))$. Using what we just have shown to T^*, we get that $T^{**} \in \mathrm{GL}(\mathcal{B}(X^{**}))$. Let $x \to j_x$ denote the canonical isometry from X into X^{**}. Note that for $x \in X$ and $\varphi \in X^*$, we have

$$[T^{**}(j_x)](\varphi) = (j_x \circ T^*)(\varphi) = j_x \circ \varphi \circ T = (\varphi \circ T)(x) = j_{T(x)}(\varphi).$$

Thus, $T^{**}(j_x) = j_{T(x)}$ for every $x \in X$. This shows that if we identify X with its isometric copy inside X^{**}, then the restriction of T^{**} to X is equal to T. Since T^{**} is injective, we get that T is injective.

To show that T is surjective, we first observe that $T(X)$ is dense in X. Indeed, assume (for contradiction) that there exists some $x \in X \setminus \overline{T(X)}$. By a corollary to the second Hahn-Banach separation theorem, we can find some $\psi \in X^*$ such that $\psi(x) \neq 0$ while $\psi = 0$ on $\overline{T(X)}$. We then have $T^*(\psi) = \psi \circ T = 0$, which implies that $\psi = 0$ since T^* is invertible in $\mathcal{B}(X^*)$, giving a contradiction.
Next, we observe that $T(X)$ is closed in X. Indeed, let $\{x_n\}$ be a sequence in X such that $T(x_n) \to y$ for some $y \in X$. For any $m, n \in \mathbb{N}$, we then have

$$\|x_n - x_m\| = \|(T^{**})^{-1}(T(x_n - x_m))\| \ \leq \|(T^{**})^{-1}\| \ \|T(x_n) - T(x_m)\|.$$

Since $\{T(x_n)\}$ is Cauchy, we get that $\{x_n\}$ i Cauchy. As X is complete, $x_n \to x$ for some $x \in X$, and we then get that $y = \lim_n T(x_n) = T(x)$, i.e., $y \in T(X)$, as desired.
Altogether, we get $T(X) = \overline{T(X)} = X$, i.e., T is surjective. Thus we have shown that T is bijective. By the open mapping theorem, we get that $T \in \mathrm{GL}(\mathcal{B}(X))$. This finishes the proof that (1) holds.

Let now $\lambda \in \mathbb{F}$. Using (1) we get

$$\lambda \in \mathrm{sp}_{\mathcal{A}}(T) \Leftrightarrow (\lambda I_X - T) \in \mathrm{GL}(\mathcal{B}(X)) \Leftrightarrow (\lambda I_X - T)^* \in \mathrm{GL}(\mathcal{B}(X^*))$$

$$\Leftrightarrow (\lambda I_{X_*} - T^*) \in \mathrm{GL}(\mathcal{B}(X^*)) \Leftrightarrow \lambda \in \mathrm{sp}_{\mathcal{B}}(T^*),$$

which shows that $\mathrm{sp}_{\mathcal{A}}(T) = \mathrm{sp}_{\mathcal{B}}(T^*)$.

Note: If H is a complex Hilbert space, $T \in \mathcal{B}(H)$, and T^* denotes the Hilbert space adjoint of T, then, using that the adjoint operation is conjugate-linear in this case, a similar, but simpler argument gives that

$$\mathrm{sp}(T^*) = \overline{\mathrm{sp}(T)} = \{\overline{\lambda} \mid \lambda \in \mathrm{sp}(T)\}.$$

Exercise 39

Let H denote a nontrivial complex Hilbert space and consider $\mathcal{A} := \mathcal{B}(H)$ as a Banach algebra. Let $\mathcal{B} = \{e_j\}_{j \in J}$ be an orthonormal basis for H and $f \in \ell^\infty(J)$. Set $\lambda_j := f(j) \in \mathbb{C}$ for each $j \in J$. Let $D \in \mathcal{A}$ denote the associated "diagonal" operator satisfying that $D(e_j) = \lambda_j \, e_j$ for every $j \in J$. We have seen in a lecture that

$$\mathrm{sp}_{\mathcal{A}}(D) = \overline{\{\lambda_j \mid j \in J\}} = \overline{f(J)}.$$

It follows that $r_{\mathcal{A}}(D) = \|D\| = \|f\|_\infty$:

The fact that $\|D\| = \|f\|_\infty$ should be known from a previous course (and is easy to show). The equality $r_{\mathcal{A}}(D) = \|f\|_\infty$ could be deduced from Exercise 35, but a direct poof goes like this. Using that $\mathrm{sp}_{\mathcal{A}}(D) = \overline{\{\lambda_j \mid j \in J\}}$ we get

$$\|f\|_\infty = \sup\{|f(j)| : j \in J\} = \sup\{|\lambda_j| : j \in J\} = \sup\{|\lambda| : \lambda \in \mathrm{sp}_{\mathcal{A}}(D)\} = r_{\mathcal{A}}(D) \leq \|D\| \ = \|f\|_\infty,$$

and the asserted equality follows.

Exercise 40

Consider the Banach algebra $\mathcal{A} = M_2(\mathbb{C}) \simeq \mathcal{B}(\mathbb{C}^2)$. *An example of a matrix* $A \in \mathcal{A}$ *satisfying that* $r_{\mathcal{A}}(A) < \|A\|$ *is as follows*:

Set $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. Then one easily checks that $\mathrm{sp}(A) = \{0\}$ and $\|A\| = 1$. So we get that

$$r_{\mathcal{A}}(A) = 0 < 1 = \|A\|.$$

Exercise 42

Let $\mathcal{A}$ denote a complex unital Banach algebra with unit $1_{\mathcal{A}}$ satisfying $\|1_{\mathcal{A}}\| = 1$. Let $a \in \mathcal{A}$ and let f be a complex polynomial given by $f(z) = \sum_{k=0}^{n} c_k z^k$ for some $c_0, c_1, \ldots, c_n \in \mathbb{C}$.

We can then define $f(a) \in \mathcal{A}$ by $f(a) := \sum_{k=0}^{n} c_k a^k$. It follows from a lemma proved in a lecture that

$$f(\mathrm{sp}_{\mathcal{A}}(a)) \subseteq \mathrm{sp}_{\mathcal{A}}\big(f(a)\big).$$

We also have that $\mathrm{sp}_{\mathcal{A}}\big(f(a)\big) \subseteq f(\mathrm{sp}_{\mathcal{A}}(a))$, *hence that* $f(\mathrm{sp}_{\mathcal{A}}(a)) = \mathrm{sp}_{\mathcal{A}}\big(f(a)\big)$:

If $n = 0$, i.e., $f(z) = c_0$ for all $z \in \mathbb{C}$, then we have $f(a) = c_0\, 1_A$, so

$$f(\mathrm{sp}_{\mathcal{A}}(a)) = c_0 = \mathrm{sp}_{\mathcal{A}}\big(f(a)\big).$$

Suppose now that $n \geq 1$. We may also suppose that $c_n \neq 0$.

Assume that $\lambda \in \mathbb{C} \setminus f(\mathrm{sp}_{\mathcal{A}}(a))$. It suffices to show that $\lambda \in \mathbb{C} \setminus \mathrm{sp}_{\mathcal{A}}(f(a))$.
Consider the polynomial q of order n given by $q(z) = \lambda - f(z)$ for all $z \in \mathbb{C}$. By the fundamental theorem of algebra, we may then write $q(z) = \alpha(\alpha_1 - z) \cdots (\alpha_n - z)$ for some $\alpha, \alpha_1, \ldots, \alpha_n \in \mathbb{C}$ with $\alpha \neq 0$. We note that

$$(*) \quad \lambda 1_{\mathcal{A}} - f(a) = q(a) = \alpha\,(\alpha_1 1_A - a) \cdots (\alpha_n 1_{\mathcal{A}} - a).$$

Let $j \in \{1, \ldots, n\}$. Since $q(\alpha_j) = 0$, we have that $f(\alpha_j) = \lambda \notin f(\mathrm{sp}_{\mathcal{A}}(a))$. This implies that $\alpha_j \notin \mathrm{sp}_{\mathcal{A}}(a)$, i.e., $(\alpha_j 1_{\mathcal{A}} - a) \in \mathrm{GL}(\mathcal{A})$. Since $\mathrm{GL}(\mathcal{A})$ is a group w.r.t. multiplication, and $\alpha \neq 0$, we obtain that $\alpha(\alpha_1 1_{\mathcal{A}} - a) \cdots (\alpha_n 1_{\mathcal{A}} - a) \in \mathrm{GL}(\mathcal{A})$.
Using $(*)$, we can now conclude that $\lambda 1_{\mathcal{A}} - f(a) \in \mathrm{GL}(\mathcal{A})$, i.e., $\lambda \in \mathbb{C} \setminus \mathrm{sp}_{\mathcal{A}}(f(a))$, as desired.