30$^{\text{th}}$ April, 2024

MAT4450 – Spring 2024

Mandatory assignment - Solution outline

Problem 1 [20 points]

Let M be a closed subspace of a locally convex Hausdorff space (X, τ) (over $\mathbb{F}$) and let $\psi : M \to \mathbb{F}$ be a continuous linear functional. Show that there exists some $\varphi \in (X, \tau)^*$ which extends ψ, that is, such that $\varphi_{|M} = \psi$.

Solution: We follow the given hint. If $\psi = 0$, the assertion is trivial. So we may assume that $\psi \neq 0$. We may then pick $y_0 \in M$ such that $\psi(y_0) \neq 0$. Setting $x_0 = \frac{1}{\psi(y_0)}\, y_0$, we have $x_0 \in M$ and $\psi(x_0) = 1$.

Set $N := \{x \in M : \psi(x) = 0\}$. Then N is a subspace of M, hence of X. Moreover, N is closed in X (since $N = \ker(\psi)$ is closed in M).

As $x_0 \notin \ker(\psi)$, we have $x_0 \notin N$. We may therefore use one of the corollaries to the Hahn-Banach separation theorem to obtain that there exists some $\varphi' \in (X, \tau)^*$ such that $\varphi'(x_0) \neq 0$ and $\varphi' = 0$ on N.
Set $\varphi := \frac{1}{\varphi'(x_0)}\varphi'$. Then $\varphi \in (X, \tau)^*$, $\varphi(x_0) = 1$ and $\varphi = 0$ on N.

Now, consider $x \in M$. Then

$$\psi(x - \psi(x)\, x_0) = \psi(x) - \psi(x)\, \psi(x_0) = \psi(x) - \psi(x) = 0,$$

so $x - \psi(x)\, x_0 \in N$. As $x = (x - \psi(x)\, x_0) + \psi(x)\, x_0$ and $\varphi|N = 0$, we get that

$$\varphi(x) = \varphi(\psi(x)x_0) = \psi(x)\, \varphi(x_0) = \psi(x).$$

Thus, φ agrees with ψ on M, as desired.

Problem 2 [20 points]

Let H be a complex Hilbert space $\neq \{0\}$. Set $B = \{\xi \in H : \|\xi\| \leq 1\}$ and $\mathcal{B} = \{T \in \mathcal{B}(H) : \|T\| \leq 1\}$.

a) *Show that* $\mathrm{ex}(B) = \{\eta \in B : \|\eta\| = 1\}$.

Solution: We first show that $\{\eta \in B : \|\eta\| = 1\} \subseteq \mathrm{ex}(B)$.

Let $\eta \in B$, $\|\eta\| = 1$. Assume $\eta = (1 - t)\, \xi + t\, \xi'$ for $\xi, \xi' \in B$ and $0 < t < 1$. Then we must have $\|\xi\| = \|\xi'\| = 1$ (otherwise, using the triangle inequality, we would get $\|\eta\| < 1$).

Observe now that if $\mathrm{Re}\, \langle \xi, \xi' \rangle < 1$, then we get that

$$\begin{aligned} 1 = \|\eta\|^2 = \|(1-t)\, \xi + t\, \xi\|^2 &= (1-t)^2 + 2t(1-t)\, \mathrm{Re}\, \langle \xi, \xi' \rangle + t^2 \\ &< (1-t)^2 + 2t(1-t) + t^2 = 1\,, \end{aligned}$$

a contradiction. So we must have $\mathrm{Re}\, \langle \xi, \xi' \rangle \geq 1$.

Using the Cauchy-Schwarz inequality, we then get that

$$1 \leq \mathrm{Re}\, \langle \xi, \xi' \rangle \leq |\langle \xi, \xi' \rangle| \leq \|\xi\|\, \|\xi'\| = 1\,,$$

thus, $\operatorname{Re}\langle \xi, \xi' \rangle = 1$ (in fact, $\langle \xi, \xi' \rangle = 1$). So

$$\|\xi - \xi'\|^2 = \|\xi\|^2 - 2\operatorname{Re}\langle \xi, \xi' \rangle + \|\xi'\|^2 = 1 - 2 + 1 = 0\,,$$

hence $\xi = \xi'$. This shows that $\eta \in \operatorname{ex}(B)$. Hence the asserted inclusion holds.

Next, we show that $\operatorname{ex}(B) \subseteq \{\eta \in B\,;\, \|\eta\| = 1\}$ holds:

Assume $\eta \in B$, $\|\eta\| \neq 1$, so $\|\eta\| < 1$. It suffices to show that $\eta \notin \operatorname{ex}(B)$.

• If $\eta = 0$, then, since $0 = \frac{1}{2}\,\xi + \frac{1}{2}\,(-\xi)$ and $\xi \neq -\xi$ for any $\xi \in B \setminus \{0\}$, we get that $\eta = 0 \notin \operatorname{ex}(B)$.

• Suppose now that $\eta \neq 0$. Set $\xi := \frac{1}{\|\eta\|}\eta$ and $t := \|\eta\|$. Then $0 \in B$, $\xi \in B$, $0 < t < 1$, and

$$(1-t)\,0 + t\,\xi \;=\; \|\eta\|\,\frac{1}{\|\eta\|}\,\eta \;=\; \eta\,.$$

As $\xi \neq 0$, this implies that $\eta \notin \operatorname{ex}(B)$.

b) *Let $T \in \mathcal{B}(H)$. Assume that T or T^* is isometric. (By T^* we mean here the adjoint operator of T as defined for a bounded operator on a Hilbert space.) Show that $T \in \operatorname{ex}(\mathcal{B})$.*

Solution: Assume first that T is an isometry. We note that $T \in \mathcal{B}$ (since an isometry is norm-preserving). Let $R, S \in \mathcal{B}$ and $0 < t < 1$ be such that

$$T = (1-t)R + tS\,.$$

We want to show that $R = S$. By linearity of R and S, it suffices to show that $R(\eta) = S(\eta)$ for every $\eta \in H$ such that $\|\eta\| = 1$. So let $\eta \in H, \|\eta\| = 1$. Since T is norm-preserving, we have $\|T(\eta)\| = \|\eta\| = 1$. Using a), we get that $T(\eta) \in \operatorname{ex}(B)$. Moreover, we have that

$$T(\eta) = (1-t)\,R(\eta) + t\,S(\eta)\,, \text{ and } \; R(\eta),\, S(\eta) \in B \tag{1}$$

because $\|R(\eta)\| \leq \|R\|\,\|\eta\| \leq 1$ and $\|S(\eta)\| \leq \|S\|\,\|\eta\| \leq 1\,.$

Since $T(\eta) \in \operatorname{ex}(B)$, we conclude from (1) that $R(\eta) = S(\eta)$, as desired. It follows $T \in \operatorname{ex}(\mathcal{B})$.

Assume now that T^* is an isometry (T is then often called a *co-isometry*). Again, let $R, S \in \mathcal{B}$ and $0 < t < 1$ be such that $T = (1-t)R + tS$. Then we get that

$$T^* = \big((1-t)R + tS\big)^* = (1-t)\,R^* + t\,S^* \tag{2}$$

and $\|R^*\| = \|R\| \leq 1$, $\|S^*\| = \|S\| \leq 1$, so $R^*, S^* \in \mathcal{B}$.

Since T^* is an isometry, we know from the first part that $T^* \in \operatorname{ext}(\mathcal{B})$. Hence it follows from (2) that $R^* = S^*$. Thus we get that $R = (R^*)^* = (S^*)^* = S$. This shows that $T \in \operatorname{ex}(\mathcal{B})$, as desired.

Problem 3 (= Exercise 4.1.6 in Pedersen's book) [15 points]

Let $\mathcal{A}$ be a unital complex Banach algebra, let $A \in \mathcal{A}$, and let Ω be an open subset of $\mathbb{C}$ containing $\mathrm{sp}(A)$. Show that there is an $\varepsilon > 0$ such that $B \in \mathcal{A}$ and $\|A - B\| < \varepsilon$ implies that $\mathrm{sp}(B) \subseteq \Omega$.

Solution. Assume (for contradiction) that the assertion does not hold. For every $n \in \mathbb{N}$ we can then find some $B_n \in \mathcal{A}$ such that $\|A - B_n\| < n^{-1}$ and $\mathrm{sp}(B_n)$ is not contained in Ω, i.e., $\mathrm{sp}(B_n) \setminus \Omega \neq \emptyset$. We may then pick $\lambda_n \in \mathrm{sp}(B_n) \setminus \Omega$ for every $n \in \mathbb{N}$. Then we have

$$|\lambda_n| \leq \|B_n\| \leq \|B_n - A\| \ + \|A\| \leq \frac{1}{n} + \|A\| \leq \|A\| + 1$$

for every n, so the sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{C}$. By the Bolzano-Weiertrass theorem, this sequence has a convergent subsequence. Hence, by passing to a subsequence if necessary, we may assume that $\{\lambda_n\}_{n\in\mathbb{N}}$ is convergent, say $\lim_{n\to\infty} \lambda_n = \lambda$ for some $\lambda \in \mathbb{C}$.

Since $\{\lambda_n\}_{n\in\mathbb{N}} \subseteq \mathbb{C} \setminus \Omega$ and $\mathbb{C} \setminus \Omega$ is closed, we get that $\lambda \in \mathbb{C} \setminus \Omega$. Since $\mathrm{sp}(A) \subseteq \Omega$ (by assumption), this implies that $\lambda \notin \mathrm{sp}(A)$, i.e., $\lambda I - A \in \mathrm{GL}(\mathcal{A})$. But $\lambda_n I - B_n$ converges to $\lambda I - A$, and $\mathrm{GL}(\mathcal{A})$ is is an open subset of $\mathcal{A}$, so we can find some $N \in \mathbb{N}$ such that $\lambda_N I - B_N \in \mathrm{GL}(\mathcal{A})$, i.e., $\lambda_N \notin \mathrm{sp}(B_N)$. But $\lambda_N \in \mathrm{sp}(B_N)$, so this gives a contradiction.

Problem 4 [20 points]

Solve Exercise 4.3.1 in Pedersen's book.

Solution. Let X and Y be compact Hausdorff spaces. Set

$$\mathcal{A} := \mathrm{span}\Big(\{f \otimes g : f \in C(X), g \in C(Y)\}\Big),$$

where $f \otimes g \in C(X \times Y)$ is defined by $(f \otimes g)(x, y) := f(x)g(y)$ for all $(x, y) \in X \times Y$. It is straightforward to verify that $\mathcal{A}$ is a self-adjoint subalgebra of $C(X \times Y)$ which contains the constant functions and separates the points of $X \times Y$. Hence it follows from the (complex) Stone-Weierstrass theorem that $\mathcal{A}$ is dense in $C(X \times Y)$ (w.r.t. the uniform norm).

Problem 5 [20 points]

Let $N \in \mathbb{N}$ and let $\mathbb{Z}_N = \{0, 1, \ldots, N-1\}$ denote the cyclic group of order N (addition being defined modulo N). Set $\mathcal{A} = \ell^1(\mathbb{Z}_N, \mathbb{C})$. It can easily be checked that $\mathcal{A}$ becomes a commutative unital Banach algebra w.r.t. the convolution product given by

$$(f * g)(n) = \sum_{m\in\mathbb{Z}_N} f(m)g(n-m) \quad \textit{for } f, g \in \mathcal{A}, n \in \mathbb{Z}_N.$$

Describe the character space $\widehat{\mathcal{A}}$ and the Gelfand transform $\Gamma : \mathcal{A} \to C(\widehat{\mathcal{A}})$ the best you can.

Solution. One proceeds in the same way as when dealing with the infinite cyclic group $\mathbb{Z}$, but there is one important difference: one now gets that $\widehat{\mathcal{A}}$ can be identified with $\mathbb{T}_N := \{\lambda \in \mathbb{T} : \lambda^N = 1\}$.

Indeed, if $\gamma \in \widehat{\mathcal{A}}$, then using that $\delta_1 * \delta_1 * \cdots * \delta_1$ (N-times) $= \delta_N = \delta_0 = I$ and that γ is multiplicative, one gets that $\gamma(\delta_1)^N = \gamma(I) = 1$, i.e., $\lambda := \gamma(\delta_1) \in \mathbb{T}_N$. It then follows that $\gamma(f) = \sum_{n\in\mathbb{Z}_N} f(n)\lambda^n$ for all $f \in \mathcal{A}$.

On the other hand, if $\lambda \in \mathbb{T}_N$, then one checks that the map $\gamma_\lambda : \mathcal{A} \to \mathbb{C}$, given by $\gamma_\lambda(f) := \sum_{n\in\mathbb{Z}_N} f(n)\lambda^n$ for all $f \in \mathcal{A}$, belongs to $\widehat{\mathcal{A}}$. The proof that γ_λ is multiplicative does require that $\lambda \in \mathbb{T}_N$ (i.e., it does not hold when $\lambda \in \mathbb{T} \setminus \mathbb{T}_N$).

To see this, let $n, m \in \mathbb{Z}_N$, and $p \in \mathbb{Z}_N$ be given by $p := n - m$ (modulo N). Then we have that $\lambda^p = \lambda^{n-m}$ (where $n - m \in \mathbb{Z}$ has its usual meaning): if $0 \le m \le n \le N-1$, this is obvious, while if $0 \le n < m \le N-1$, then

$$\lambda^p = \lambda^{N+n-m} = \lambda^N \lambda^{n-m} = \lambda^{n-m}.$$

This implies that

$$\lambda^m \lambda^p = \lambda^m \lambda^{n-m} = \lambda^n,$$

and the same computation as for $\mathbb{Z}$ is easily seen to go through by making use of this formula.

Identifying $\mathbb{T}_N$ with $\widehat{\mathcal{A}}$ via the map $\gamma \mapsto \gamma_\lambda$, the Gelfand transform Γ of $\mathcal{A}$ becomes the map $\Gamma : \mathcal{A} \to C(\mathbb{T}_N)$ given by

$$[\Gamma(f)](\lambda) = \sum_{n\in\mathbb{Z}_N} f(n)\lambda^n, \quad \lambda \in \mathbb{T}_N.$$

Set now $\omega := e^{i\,2\pi/N}$, so that $\mathbb{T}_N = \{\omega^0, \omega^1, \omega^2, \ldots, \omega^{N-1}\}$.

Note that the map $h \mapsto (h(\omega^0), h(\omega^1), \ldots, h(\omega^{N-1}))$ is an isometric isomorphism from the Banach algebra $C(\mathbb{T}_N)$ onto the Banach algebra $\mathbb{C}^N$ (with the $\|\cdot\|_\infty$-norm and the pointwise product).

On the other hand, $\mathcal{A} = \ell^1(\mathbb{Z}_N, \mathbb{C})$ can also be identified as a Banach algebra with $\mathbb{C}^N$ (but now with the $\|\cdot\|_1$-norm and the convolution product), via the map $f \mapsto (f(0), f(1), \ldots, f(N-1)))$.

Using these identifications, the Gelfand transform of $\mathcal{A}$ corresponds to the map $\Gamma : \mathbb{C}^N \to \mathbb{C}^N$ given by

$$\Gamma(z_0, z_1, \ldots, z_{N-1}) = \Big(\sum_{n=0}^{N-1} z_n, \sum_{n=0}^{N-1} \omega^n z_n, \sum_{n=0}^{N-1} \omega^{2n} z_n, \cdots, \sum_{n=0}^{N-1} \omega^{(N-1)n} z_n\Big),$$

whose standard matrix is the (unitary) matrix

$$F_N := \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{N-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \omega^{N-1} & \omega^{2(N-1)} & \cdots & \omega^{(N-1)^2} \end{bmatrix},$$

i.e.,

$$F_N = \left[e^{i\, jk\, 2\pi/N} \right]_{j,\,k \in \{0,1,\ldots,N-1\}}.$$

This matrix is frequently called the $N \times N$ *Fourier matrix.*