19th March, 2024

MAT4450 – Spring 2024

Mandatory assignment 1 of 1

Submission deadline

Thursday 18th of April 2024, 14:30 in Canvas (canvas.uio.no).

Instructions

Note that you have **one attempt** to pass the assignment. This means that there are no second attempts.

You can choose between scanning handwritten notes or typing the solution directly on a computer (for instance with LaTeX). The assignment must be submitted as a single PDF file. Scanned pages must be clearly legible. The submission must contain your name and the course number.

It is expected that you give a clear presentation with all necessary explanations. All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. If we doubt that you have understood the content you have handed in, we may request that you give an oral account.

To get a passing grade, at least one half of the exercise set must be answered in a satisfactory way.

Application for postponed delivery

If you need to apply for a postponement of the submission deadline due to illness or other reasons, you have to contact the Student Administration at the Department of Mathematics (e-mail: studieinfo@math.uio.no) no later than the same day as the deadline.

All mandatory assignments in this course must be approved in the same semester, before you are allowed to take the final examination.

Complete guidelines about delivery of mandatory assignments:

uio.no/english/studies/admin/compulsory-activities/mn-math-mandatory.html

GOOD LUCK!

Problem 1 [20 points]

Let M be a closed subspace of a locally convex Hausdorff space (X, τ) (over $\mathbb{F}$) and let $\psi : M \to \mathbb{F}$ be a continuous linear functional. Show that there exists some $\varphi \in (X, \tau)^*$ which extends ψ, that is, such that $\varphi_{|M} = \psi$.

Hint: *Consider* $N = \{x \in M : \psi(x) = 0\}$. *If* $\psi \neq 0$, *then one may find* $x_0 \in M$ *such that* $\psi(x_0) = 1$, *so* $x_0 \in X \setminus N$. *Use one version of the Hahn-Banach separation theorem.*

Problem 2 [20 points]

Let H be a complex Hilbert space $\neq \{0\}$. Set $B = \{\xi \in H \,:\, \|\xi\| \leq 1\}$ and $\mathcal{B} = \{T \in \mathcal{B}(H) \,:\, \|T\| \leq 1\}$.

We recall that if C is a convex subset of a vector space, then $\mathrm{ex}(C)$ denotes the set of all extreme points of C.

a) Show that $\mathrm{ex}(B) = \{\eta \in B \,:\, \|\eta\| = 1\}$.

b) Let $T \in \mathcal{B}(H)$. Assume that T or T^* is isometric. (By T^* we mean here the adjoint operator of T as defined for a bounded operator on a Hilbert space.) Show that $T \in \mathrm{ex}(\mathcal{B})$.

Problem 3 [20 points]

Solve Exercise 4.1.6 in Pedersen's book.

Problem 4 [20 points]

Solve Exercise 4.3.1 in Pedersen's book.

Problem 5 [20 points]

Let $N \in \mathbb{N}$ and let $\mathbb{Z}_N = \{0, 1, \ldots, N-1\}$ be the cyclic group of order N (addition being defined modulo N). Consider the Banach space $\mathcal{A} = \ell^1(\mathbb{Z}_N, \mathbb{C})$ with the $\|\cdot\|_1$-norm. It can easily be checked that $\mathcal{A}$ becomes a commutative unital Banach algebra w.r.t. the convolution product given by

$$(f * g)(n) := \sum_{m \in \mathbb{Z}_N} f(m) g(n-m) \quad \text{for each } n \in \mathbb{Z}_N,$$

for all $f, g \in \mathcal{A}$. Describe the character space $\widehat{\mathcal{A}}$ of $\mathcal{A}$ and the Gelfand transform $\Gamma : \mathcal{A} \to C(\widehat{\mathcal{A}})$ as best you can.