Mandatory assignment: MAT4500, Fall 2016.
Deadline: 2:30 pm, 7th floor, N. H. Abel's house, October 21, 2016.

Read the information concerning this assignment available at

$$http://www.uio.no/studier/emner/matnat/math/MAT4500/index-eng.html$$

(It is recommended that the mandatory assignment be written in latex, but legible handwritten solutions are also fine.) Show all your work. Good luck!

Exercise 1. Homeomorphisms and products

a) Let X and Y be topological spaces and let $\tau\colon X \times Y \to Y \times X$ be the "twist map" that sends (x, y) to (y, x). Show that τ is a homeomorphism.

b) Let X be a topological space and let $\tau\colon X \times X \times X \to X \times X \times X$ be the "cyclic permutation map" that sends (x, y, z) to (z, x, y). Show that τ is a homeomorphism.

c) Let $\mathscr{A}$ be a nonempty set and suppose $f_\alpha\colon X_\alpha \to Y_\alpha$ is a homeomorphism for each $\alpha \in \mathscr{A}$. Show that the function

$$\prod_{\alpha\in\mathscr{A}} f_\alpha\colon \prod_{\alpha\in\mathscr{A}} X_\alpha \to \prod_{\alpha\in\mathscr{A}} Y_\alpha$$

is a homeomorphism.

Exercise 2. Dense subsets and functions

A subset A of a topological space X is called dense in X if $\overline{A} = X$.

a) Show that $A \subset X$ is dense if and only if every nonempty open subset U of X intersects A.

b) If U and V are open dense subsets of X, show that $U \cap V$ is dense in X.

c) Suppose $f, g\colon X \to Y$ are continuous functions to a Hausdorff space Y, and $A \subset X$ is a dense subset. If the restrictions of f and g to A agree, i.e., $f_{|A} = g_{|A}$, show that $f = g$.

d) Suppose $f\colon \mathbf{R} \to \mathbf{R}$ is a function such that $f(x + y) = f(x) + f(y)$ for all real numbers $x, y \in \mathbf{R}$. If f is continuous, show that there exists a real number c such that $f(x) = cx$ for all $x \in \mathbf{R}$.

e) [Bonus question, solution not required.] Is the conclusion in **c**) valid if f is not continuous? [Hint: $\mathbf{R}$ is a $\mathbf{Q}$-vector space, and every vector space over a field has a (possibly infinite) basis by Zorn's lemma.]

Exercise 3. Quotient spaces

Let Q be the set of all planes through the origin in $\mathbf{R}^3$. To each unit vector $\underline{x} = (x_1, x_2, x_3)$ in $\mathbf{R}^3$ we associate the plane

$$q(\underline{x}) = \{\underline{y} \in \mathbf{R}^3 \mid \Sigma x_i y_i = 0\}.$$

This furnishes a surjective function $q\colon S^2 \to Q$, where S^2 denotes the unit 2-sphere. Give Q the quotient topology induced by q.

a) Show that q is an open map.

b) Show that Q is homeomorphic to the projective plane $\mathbf{P}^2$.

c) Let $H \subset Q \times \mathbf{R}^3$ be the set of all pairs $(\mathscr{P}, \underline{y})$, where $\underline{y} \in \mathscr{P}$. Show that H is a closed subset in the product space $Q \times \mathbf{R}^3$.