1.5 MAT4500 26.08.16

(a) $\Longleftrightarrow$

(b) $\Longleftarrow$

(c) $\Longrightarrow$

(d) $\Longleftrightarrow$

(a)

Note: $x \in \bigcup_{A \in \mathcal{A}} A \Longleftrightarrow \exists A \in \mathcal{A}$ s.t. $x \in A$

$\Updownarrow$ ~~negation~~ Contrapositive

$x \in \bigcap_{A \in \mathcal{A}} A \Longleftrightarrow \forall A \in \mathcal{A} \quad x \in A$

(d)

~~i.e.~~

i.e. $\cup \longleftrightarrow \cap$ when you take the contrapositive of a statement, in the same way as $\exists \longleftrightarrow \forall$.

1.7

$D = A \cap (B \cup C)$

$E = (A \cap B) \cup C$

$F = (A \cap B \cap C) \cup (A - B)$

For F there are two cases:

$x \in B \Rightarrow x \in C \Rightarrow x \in B \cap C \cap A$

$x \notin B \Rightarrow x \in A - B$

1.8

$A = \{0, 1\} \Rightarrow P(A) = \{\emptyset, \{0\}, \{1\}, \{0,1\}\}, \# = 4$

$A = \{0\} \Rightarrow P(A) = \{\emptyset, \{1\}\}, \# = 2$

$A = \{0, 1, 2\} \Rightarrow \# = 8$

$A = \emptyset \Rightarrow \# = 1, P(A) = \{\emptyset\}$

In general $P(A)$ has $2^{|A|}$ elements when $A$ is finite (this do also make sense for $|A|$ infinite).

2

2.3

(f) Q: $f\left(\bigcup_{A\in\mathcal{A}} A\right) = \bigcup_{A\in\mathcal{A}} f(A)$

$y \in f\left(\bigcup_{A\in\mathcal{A}} A\right) \Leftrightarrow \exists\ x\ \bigcup_{A\in\mathcal{A}} A$ s.t. $f(x)=y$

$\Leftrightarrow\ \exists\, A \in \mathcal{A}$ s.t. $\exists\, x \in A$ s.t. $f(x)=y$

$\Leftrightarrow\ \exists\, A \in \mathcal{A}$ s.t. $y \in f(A)$

$\Leftrightarrow\ y \in \bigcup_{A\in\mathcal{A}} f(A)$

(c) Q: $f^{-1}\left(\bigcap_{A\in\mathcal{A}} A\right) = \bigcap_{A\in\mathcal{A}} f^{-1}(A)$

$x \in f^{-1}\left(\bigcap_{A\in\mathcal{A}} A\right) \Leftrightarrow f(x) \in \bigcap_{A\in\mathcal{A}} A$

$\Leftrightarrow \forall\, A \in \mathcal{A},\ f(x) \in A$

$\Leftrightarrow \forall\, A \in \mathcal{A}\quad x \in f^{-1}(A)$

$\Leftrightarrow\ x \in \bigcap_{A\in\mathcal{A}} f^{-1}(A)$

(b) & (g) are similar.

§3.1 Q: Equivalence relation on $\mathbb{R}^2$:

$$(x_0, y_0) \sim (x_1, y_1) \text{ iff } y_0 - x_0^2 = y_1 - x_1^2$$

1. Check that this is an equivalence relation.
2. What are the equivalence classes?

A:
1. Reflexive $(x, y) \sim (x, y)$ since $y - x^2 = y - x^2$

Symmetric: $y_0 - x_0^2 = y_1 - x_1^2 \iff y_1 - x_1^2 = y_0 - x_0^2$

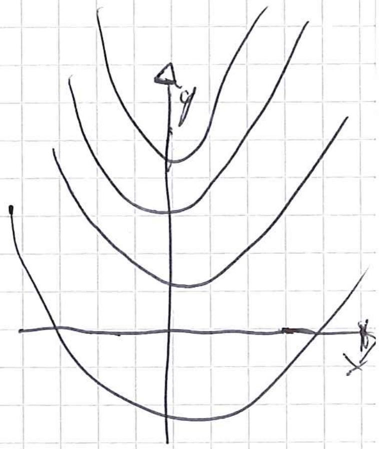
Reflexive: $(x_0, y_0) \sim (x_1, y_1) \sim (x_2, y_2)$

$\iff y_0 - x_0^2 = y_1 - x_1^2 = y_2 - x_2^2 \Rightarrow y_0 - x_0^2 = y_2 - x_2^2$

$\iff (x_0, y_0) \sim (x_2, y_2)$

2. ~~Let [(x_0, y~~

The equiv class of $(x_0, y_0)$ is:




$\overline{(x_0,y_0)} := \{ (x, y) \mid (x, y) \sim (x_0, y_0) \}$

$= \{ (x, y) \mid y - x^2 = y_0 - x_0^2 \}$

= Parabola passing through $(x_0, y_0), (-x_0, y_0)$ & $(0, y_0 - x_0^2)$

The set of equiv classes is in bijection with $\mathbb{R}$, e.g. $\mathbb{R} \ni c \longmapsto \overline{(0, c)}$

Parametrizes the equiv. classes / moduli space.

3.3: Q: $\sim$ relation, symmetric & transitive

What is wrong with: $a \sim b \overset{\text{symmetric}}{\Rightarrow} b \sim a$,

$a \sim b$ & $b \sim a \Rightarrow a \sim a \Rightarrow \sim$ is reflexive.

A: We do not know if $b$ exists. I.e. given $a$ there might be no $b$ s.t. $a \sim b$. (Reflexivity fixes this.)

3.15(a) Assume $\mathbb{R}$ has the least upper bound property (LUB)

Show: $[0,1]$ & $[0,1)$ have LUB.

Proof for $[0,1)$: Consider a subset $A_0 \subseteq [0,1)$ which is bounded above by $X$.

$\Rightarrow \forall a \in A_0 \quad 0 \leq a \leq X < 1$

~~By~~ Assumption on $\mathbb{R} \Rightarrow \exists\ y$ s.t. $0 \leq a \leq y \ \forall a \in A_0$ & $y \leq X \Rightarrow y \in [0,1) \Rightarrow [0,1)$ has LUB.

Proof is similar for $[0,1]$.

(b): Which of the following have LUB?

1. $[0,1] \times [0,1]$ in dictionary order. Yes
2. $[0,1] \times [0,1)$ —— " ——. No
3. $[0,1) \times [0,1]$ —— " ——. Yes

1. Consider the projection $pr_1 : [0,1] \times [0,1] \to [0,1]$, $(x,y) \mapsto x$

Let $A \subseteq [0,1] \times [0,1]$ be bounded by $(x_0, y_0)$.
Then $pr_1(A) \subseteq [0,1]$ is bounded by $pr_1(x_0,y_0) = x_0 \Rightarrow pr_1(A)$ has LUB $x_1$

Two cases: (A) $(x_1, 0) \geq (a,b) \quad \forall (a,b) \in A$.
$\Rightarrow (x_1, 0)$ is LUB for $A$.

(B) $\exists\ b > 0$ s.t. $(x_1, b) \in A$

consider $pr_2(\{(x_1,b) \in A\}) =: B$ is bounded by $1$, $\Rightarrow \exists\ y_1$ LUB for $B$

$\Rightarrow (x_1, y_1)$ is LUB for $A$

"Indeed, if $(x_2,y_2) < (x_1,y_1)$ was a smaller bound, then $x_2 = x_1$, but $y_2 < y_1$ contradicts $y_1$ being a LUB."

3. Same proof works.

3.15(b) 2. The set $\{(0, 1-\frac{1}{n+1}) \mid n \in \mathbb{N}\} =: A$ is bounded by $(1, \frac{1}{2})$, but has no LUB.
Indeed, assume $(x_1, y_1)$ is a LUB for $A$.
Then $(x_{1/2}, 0)$ is strictly smaller and bounds $A$.
(Note $x_1 > 0$, if not you could pick $n$ s.t. $y_1 < 1-\frac{1}{n+1}$, but then $(x_1, y_1)$ does not bound $A$).

1.4 (a) ~~...~~ Show using induction that any subset $\neq \emptyset$ of $\{1, \dots n\}$, $n \in \mathbb{Z}_+$ has a largest element.

Proof: True for $n=1$. (There is one ~~nonzero~~ $\neq \emptyset$ subset!)

Assume true for $n-1$.

Consider $A \subseteq \{1 \dots n\}$

Two cases:

$n \notin A$: $\Rightarrow A \subseteq \{1 \dots n-1\} \Rightarrow$ ok by induction

$n \in A$: $\Rightarrow n$ is the largest element

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(b) Q: Why can't you use this to prove that any subset of $\mathbb{Z}_+$ has a smallest element?

A: The proof of Thm 4.1 breaks down. Indeed $\mathbb{Z}_+$ itself provides a counterexample, after all $\mathbb{Z}_+$ is unbounded.

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6.3 Q: Find bijection between $\{0,1\}^\omega$ and a proper subset of itself:

A: Recall $\{0,1\}^\omega = \prod_{i \in \mathbb{Z}_+} \{0,1\} = X^\omega$

Define $f: X^\omega \to X^\omega$ by $f(x_1, x_2 \dots x_n, \dots)$

$= (0, x_1, x_2, \dots x_n, \dots)$ (Eilenberg swindle)

6.6(a) Let $A = \{1 \dots n\}$, find a bijection of $P(A)$ with $X^n$, $X = \{0,1\}$.

Proof: Recall $P(A) = \{B \subseteq A\}$, define $f: P(A) \to X^n$

$f(B) = (x_1, \dots, x_n)$ $x_i = \begin{cases} 0 & i \notin B \\ 1 & i \in B \end{cases}$. Inverse: $g: X^n \to P(A)$

$g(x_1 \dots x_n) = \{i \mid x_i = 1\}$.

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6.6 (b) Show $P(A)$ is finite if $A$ is finite.

Proof: ~~Pf~~ $A$ is in bijection with $\{1 \ldots n\}$ for some $n$. $\Rightarrow$ $P(A)$ is in bij with $P(\{1 \ldots n\})$ which is in bij. with ~~the~~ $\{0,1\}^n$, which is a finite product of finite sets, i.e finite (it is in bij with $\{1 \ldots 2^n\}$).