12.9.16 MAT4500

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6.1 (a) Q: List all injective maps $f : \{1,2,3\} \to \{1,2,3,4\}$. Show that no is bijective.

There are $4! = 24$ different injective maps, that is too many for me. If you really want to see the maps, write up the $4!$ permutations of $\{1,2,3,4\}$, and read off the first three terms of each permutation.

To show, no bijection, ~~check~~ convince yourself that there is always one element missing in the image.

(b) Q: # of injective maps $f : \{1, .. 8\} \to \{1, ..., 10\}$?

A: $\dfrac{10!}{2!} \sim \cancel{8 \cdot 10^5} \; 2 \cdot 10^6$, one way to see this: 1 can map to 1 .. 10, 2 can map to 9 different numbers, 3 can map to 8 different numbers and so on . . .

6.3
6.6 } We did these exercises last time.

7.1 Q: Prove $\mathbb{Q}$ is countably infinite.

Proof: $\mathbb{Z}_+ \subset \mathbb{Q}$, so $\mathbb{Q}$ is infinite.

Take an element $\frac{a}{b} \in \mathbb{Q}$, assume a & b are relatively prime. Consider $a1b$ as ~~a~~ number written in base 12 as follows: Write a & b ~~in~~ base 10; 1, 2 ... 9 are the same digits, 0 is the 10'th digit & 1 is the 11'th digit. This defines an injection $\mathbb{Q} \to \mathbb{Z}_+$, so $\mathbb{Q}$ is countably infinite.

(Of course many alternative, perhaps easier, proofs can be given.)

7.3 Q: Find an ~~bi~~jection $f : P(\mathbb{Z}_+) \to \{0,1\}^\omega$.

A: This is the same as 6.6(a) which we did last time, just add some extra dots.

7.4 (a) Q: Show that the set of algebraic numbers is countable.

A: Using the same approach as in 7.1 this exercise is very easy. I'll do it in a different way: We produce an F injection from the set X of algebraic numbers to $\coprod_{n \geq 0} (\mathbb{Z}^+)^{n+1}$.
(disjoint union)

The latter set is a countable union of countable sets, i.e. countable.

For any $x \in X$ pick ~~some~~ the minimal monic polynomial f (minimal of degree n) s.t. $f(x) = 0$. ~~Number~~ Order the roots of f, $x_1 < \dots < x_d$, $d \leq n$, assume $x = x_l$, $1 \leq l \leq d$.

Then define $F(x) = (\varphi(a_0), \dots \varphi(a_{n-1}), l)$,

where $f(y) = y^n + a_{n-1} y^{n-1} + \dots + a_0$, and

$\varphi : \mathbb{Q} \to \mathbb{Z}_+$ is some bijection.

Clearly $F(x)$ determines x uniquely.

(Note: We choose f to be minimal to avoid using any countable axiom of choice.)

(b) Q: Show the transcendental numbers ($= \mathbb{R} \setminus$ algebraic) are uncble, assuming the reals are.

A: $\mathbb{R}$ = Algebraic $\cup$ Transcendental. If both algebraic & transcendental numbers were countable, $\mathbb{R}$ would be cble, a contradiction.

9.5 (a) Q: $f: A \to B$ surjective $\Rightarrow \exists\, h: B \to A$ s.t.

$\cancel{gf = id_A}$ $\cancel{hf = id_A}$ $fh = id_B$

A: We use lemma 9.2. Define a collection of non-empty sets $\mathcal{B} := \{ f^{-1}(b) \mid b \in B \}$.
Here we use surjectivity to ensure $f^{-1}(b) \neq \emptyset$.
Lemma 9.2 $\Rightarrow \exists\ c: \mathcal{B} \to \bigcup_{f^{-1}(b) \in \mathcal{B}} f^{-1}(b) = A$

s.t. $c(f^{-1}(b)) \in f^{-1}(b)$.
Precompose c with the function $l: B \to \mathcal{B}$

s.t. $l(b) = f^{-1}(b)$, to obtain h, i.e. $h = cl$
Then $fh = fcl = id_B$
Indeed, $fcl(b) = fc(f^{-1}(b)) = f(a) = b$, where $a = c(f^{-1}(b)) \in f^{-1}(b)$.

(b) Q: $f: A \to B$ injective $\Rightarrow \exists\, g: B \to A$ s.t.

$gf = id_A$.

A: Write $B = f(A) \amalg (B - f(A))$

For $b \in f(A)$, we have $b = f(a)$ for some unique $a \in A$, so define $g(b) = a$.
Pick some $a_0 \in A$ (possible since $A \neq \emptyset$)
For $b \notin f(A)$ i.e. $b \in B - f(A)$ define $g(b) = a_0$.
Then g is a left inverse. Indeed,

$gf(a) = g(f(a)) = a$ by definition of g.

We did not need the axiom of choice. Note, there are no maps $A \to \emptyset$, except if $A = \emptyset$, in which case there is $id_\emptyset$.

11.8

(a) Q: A independent $v \notin \operatorname{Span} A$
$\Rightarrow A \cup \{v\}$ is independent

A: Assume $A \cup \{v\}$ is not independent
$\Rightarrow \exists v_1 \dots v_n \in A \cup \{v\}$ & $x_1 \dots x_n$ nonzero scalars s.t. $\sum x_i v_i = 0$
If $\forall v_i \in A$ then A is not independent, a contradiction.
If $v = v_i$, say for $i=1$, then $v = \frac{1}{x_1} \sum_{i \neq 1} x_i v_i$,
but then v is in span A, a contradiction.
$\Rightarrow A \cup \{v\}$ is independent.

(b) We apply Zorns lemma: Let $\mathcal{A}$ be the collection of independent sets in V. Give $\mathcal{A}$ ~~a~~ partial order by inclusion. Assume $\{T_\alpha, T_\beta \dots T_{\gamma}, \dots\}$ is a simply ordered subset of $\mathcal{A}$, i.e. we have an increasing sequence of independent sets
$T_\alpha \subseteq T_\beta \subseteq \dots \subseteq T_\gamma \subseteq \dots$
Then $\bigcup_\alpha T_\alpha$ is an independent set & $T_\delta \subseteq \bigcup_\alpha T_\alpha \ \forall \delta$.

Indeed any collection $v_1 \dots v_n \in \bigcup T_\alpha$ is in some T_j (e.g. $j = \max\{\min\{l \mid v_i \in T_l\} \mid i = 1 \dots n\}$), hence they are independent. ~~Then~~ That is $\bigcup_\alpha T_\alpha$ is an upper bound for $\{T_\alpha\}_\alpha$ & it is in $\mathcal{A}$, so Zorns lemma imply $\mathcal{A}$ has at least one maximal element.

(c) Let $A \in \mathcal{A}$ be a maximal element. If span $A \neq V$
$\Rightarrow \exists v \notin \operatorname{Span} A$, but then $A \cup \{v\}$ is independent by (a) & strictly greater than A, contradicting maximality of A. Hence span $A = V$ & A is a basis.