MAT 4500 ~~9.9~~ 9.9

13.1 Q: X top. spc. $A \subseteq X$. Assume $\forall x \in A$ $\exists$ U open s.t. $x \in U$ & $U \subseteq A$
Show A open in X.

A: $A = \bigcup_{x \in A} U_x$, where for each $x \in A$ U_x is some open set in X s.t. $x \in U_x$ & $U_x \subseteq A$. By axiom (2) of a top space A is open in X.

13.4(a) Q: $\{T_\alpha\}$ family of toplgies on X
Show $T := \bigcap_\alpha T_\alpha$ is toplgy on X.

A: We check the axioms: (1) $\emptyset, X \in T$, since $\emptyset, X \in T_\alpha$ $\forall \alpha$.

(2) Let $\{U_\beta\}_\beta \subseteq T$, is $\bigcup U_\beta \in T$?
Each $U_\beta \in T_\alpha$ $\forall \alpha \Rightarrow \bigcup U_\beta \in T_\alpha$ $\forall \alpha \Rightarrow \bigcup U_\beta \in T$.
$\Rightarrow$ (2) is OK.

(3) Let $\{U_n\}_{n=1}^{l} \subseteq T$ is $\bigcap_{n=1}^{l} U_n \in T$?
Each $U_n \in T_\alpha$ $\forall \alpha \Rightarrow \bigcap_{n=1}^{l} U_n \in T_\alpha$ $\forall \alpha \Rightarrow \bigcap_{n=1}^{l} U_n \in T$

Q: Is $\bigcup T_\alpha$ a topology on X.

A: Not necessarily. Both (2) & (3) may be violated. In ~~graph~~ 13.4(c) $T_1 \cup T_2$ doesnot satisfy ~~(2)~~ (3). If you replace one of the singletons $\{a\}$ with $\{c\}$ (2) is violated.

13.4(b) Q: $\{T_\alpha\}$ family of toplgies on X
Show $\exists!$ smallest top T_M containing all the T_α, i.e. $T_\alpha \subseteq T_M$ $\forall \alpha$
& if T' is some top on X s.t. $T_\alpha \subseteq T'$ $\forall \alpha$ then $T_M \subseteq T'$

Similarly, Show $\exists!$ T_m largest top on X contained in all T_α.

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A: Let $\mathcal{F} := \{\mathcal{T} \mid \mathcal{T}$ tplgy on X s.t. $\mathcal{T}_\alpha \subseteq \mathcal{T}\ \forall \alpha\}$

$\mathcal{F}$ is nonempty since the discrete topology is finer than any tplgy.

Define $\mathcal{T}_M := \bigcap_{\mathcal{T}\in\mathcal{F}} \mathcal{T}$. $\mathcal{T}_M$ is a topology by (a), and satisfies the minimality by definition of $\mathcal{F}$ (and properties of $\cap$). $\blacksquare$

For the second part consider $\mathcal{T}_m = \bigcap_\alpha \mathcal{T}_\alpha$, and prove that it is the largest.

13.4 (c) Q: Find $\mathcal{T}_M, \mathcal{T}_m$ for $\{\mathcal{T}_1, \mathcal{T}_2\}$, $\mathcal{T}_1 = \{\emptyset, X, \{a\}, \{a,b\}\}$, $\mathcal{T}_2 = \{\emptyset, X, \{a\}, \{b,c\}\}$.

$\mathcal{T}_M = \{\emptyset, X, \{a\}, \{a,b\}, \{b,c\}, \{b\}\}$

$\mathcal{T}_m = \{\emptyset, X, \{a\}\}$

13.5 Q: $\mathcal{A}$ (Sub) basis for top $\mathcal{T}$ on X.

Show $\mathcal{T} = \bigcap_{\mathcal{T}'\in\mathcal{F}} \mathcal{T}'$, where

$\mathcal{F} := \{\mathcal{T}' \mid \mathcal{T}' \supseteq \mathcal{A}\}$.

A: Clearly $\mathcal{T} \supseteq \mathcal{A}$ so $\mathcal{T} \supseteq \bigcap_{\mathcal{T}'\in\mathcal{T}} \mathcal{T}'$.

Claim: $\mathcal{T}' \supseteq \mathcal{A} \Rightarrow \mathcal{T}' \supseteq \mathcal{T}$.

Assume $U \in \mathcal{T}$ then U = union of finite intersections of elements in $\mathcal{A}$ $\xRightarrow{\text{by lemma 13.1/Definition}}$ $U \in \mathcal{T}' \Rightarrow \mathcal{T}' \supseteq \mathcal{T} \Rightarrow \mathcal{T} \subseteq \bigcap_{\mathcal{T}'\in\mathcal{T}} \mathcal{T}'$

$\Rightarrow \mathcal{T} = \bigcap_{\mathcal{T}'\in\mathcal{T}} \mathcal{T}'$

Beware,

13.6 Q: Show that $\mathbb{R}_\ell$ & $\mathbb{R}_K$ are not comparable,

i.e. $\mathbb{R}_\ell \not\subseteq \mathbb{R}_K$ & $\mathbb{R}_K \not\subseteq \mathbb{R}_\ell$.

Recall $\mathbb{R}_\ell$ has basis $\{[a,b) \mid a, b \in \mathbb{R}\}$

$\mathbb{R}_K$ has basis $\{(a,b)\ \&\ (a,b) - K \mid a, b \in \mathbb{R}\}$, $K = \{\frac{1}{n} \mid n \in \mathbb{Z}_+$

A: We will show $\tau_{\mathbb{R}_\ell} \ni [0,1) \notin \tau_{\mathbb{R}_K}$,

$\tau_{\mathbb{R}_K} \ni (-1,1)-K \notin \tau_{\mathbb{R}_\ell}$.

Assume $[0,1) \in \tau_{\mathbb{R}_K} \Rightarrow \exists\, B$ basis element s.t. $0 \in B$ & $B \subseteq [0,1)$. But $B=(a,b)$ or $B=(a,b)-K$ for some $a,b \in \mathbb{R}$, but if $a<0<b$, then B contains $\frac{a}{2}<0$, but $a/2 \notin [0,1)$, a contradiction.

Assume $(-1,1)-K \in \tau_{\mathbb{R}_K} \Rightarrow \exists\, B$ a basis element s.t. $0 \in B$ & $B \subseteq (-1,1)-K$. We know $B=[a,b)$ for some $a \leq 0 < b$. Pick n s.t. $\frac{1}{n}<b$. Then $\frac{1}{n} \in B$, but $\frac{1}{n} \notin (-1,1)-K$, contradicting $B \subseteq (-1,1)-K$.

16.3 Q: Let $Y=[-1,1]$ as subspace of $\mathbb{R}$.

	open Y	open $\mathbb{R}$	
A	✓	✓	$A=(-1,-\frac{1}{2}) \cup (\frac{1}{2},1)$
B	✓	✗	$B \overset{\text{e.g.}}{=} ((-2,-\frac{1}{2}) \cup (\frac{1}{2},2)) \cap Y$
C	✗	✗	$C=(-1,-\frac{1}{2}] \cup [\frac{1}{2},1)$
D	✗	✗	$D=[-1,-\frac{1}{2}] \cup [\frac{1}{2},1]$
E	✓	✓	$E=(-1,1)-K \cup \{0\}$

$K=\{\frac{1}{n} \mid n \in \mathbb{Z}_+\}$

B, C, D are proven in the same way as you show $[a,b)$ and $[a,b]$ are not open in $\mathbb{R}$.

For E, pick $x \in E \Rightarrow 1/x \notin \mathbb{Z}_+$.

If $x<0$ then $x \in (-1, x/2) \subseteq E$

If $x>0$, pick $n \in \mathbb{Z}_+$ s.t. $n<\frac{1}{x}<n+1$

Then $x \in (\frac{1}{n+1}, \frac{1}{n}) \subseteq E$.

16.4 Q: $f: X \to Y$ is open iff $\forall\, U \overset{\text{open}}{\subseteq} X$ $f(U)$ is open.
Show $\pi_1: X \times Y \to X$ & $\pi_2: X\times Y \to Y$ are open maps.

A: Assume $U \subseteq X \times Y$ is open $\Rightarrow \forall\, x\times y \in \cancel{X \times Y}$
$\exists\; V$ open $\subseteq X$, W open $\subseteq Y$ s.t. $x \times y \in V \times W \subseteq U$.
Then $\pi_1(V \times W) = V$ = open in X.

Write $U = \bigcup_{\substack{U \supseteq B \in \text{Basis} \\ \text{of } X\times Y}} B$. Then $\pi_1(U) = \pi_1(\bigcup_{B \supseteq U} B)$

$= \bigcup_{B \supseteq U} \pi_1(B)$ = union of open sets in X, i.e. open in X.

16.9 Q: Show that the dictionary order topology on $\mathbb{R}\times\mathbb{R}$ equals the product topology of $\mathbb{R}_d \times \mathbb{R}$. Discrete top on $\mathbb{R}$.

A: A basis for $\mathbb{R}_d$ is $\{\{x\} \mid x \in \mathbb{R}\}$.
We will show that $T_{\text{dictionary}} \supseteq$ Basis on $\mathbb{R}_d \times \mathbb{R}$
(we use Theorem 15.1): A basis element is of the form $\{x\} \times (a, b)$. But this is open in the dictionary order top.

Let $(a\times b, c \times d)$ be a basis element in dict. order top.
Then this equals $\{a\} \times (b, \infty) \cup (a, c) \times \mathbb{R} \cup \{c\} \times (-\infty, d)$, which is open in $\mathbb{R}_d \times \mathbb{R}$. Hence $T_{\mathbb{R}_d \times \mathbb{R}} \supseteq$ Basis of dict order top, hence $T_{\text{dict}} = T_{\mathbb{R}_d \times \mathbb{R}}$.
This is much finer than the ordinary tplgy on $\mathbb{R}\times\mathbb{R}$, since $\mathbb{R}_d$ is much finer than $\mathbb{R}$.

16.10 Q: Let $I = [0,1]$. Compare product top on $I \times I$ (T_1), dictionary order top on $I\times I$ (T_2) and top. inherited as subspace of $\mathbb{R}\times\mathbb{R}$ in dict. order top. (T_3).

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A: We have $\tau_1 \subsetneq \tau_3$

~~AHX~~

$\tau_2 \subsetneq$

~~First we prove~~ $T_1 \not\supset T_2$: The set $\{\frac{1}{2}\}\times(0,1)$ is open in T_2, but not in T_1.

$T_1 \not\subset T_2$: The set $I\times[0,1)$ is not open in T_2. Indeed, assume $\{\frac{1}{2}\times 0\}$ is contained in some basis element $B=(a\times b, c\times d)\subseteq I\times[0,1)$ i.e. $a\times b<\frac{1}{2}\times 0< c\times d$, the we must have $a<\frac{1}{2}$, but then $\frac{1}{2}(a+\frac{1}{2})\times 1\in(a\times b, c\times d)$, contradicting $B\subseteq I\times[0,1)$. (There are also other kinds of basis elems, why don't we check these?)

To prove $T_1\subset T_3$ use the same argument as in 16.9. Since T_3 is inherited from $\mathbb{R}_\ell\times\mathbb{R}$, we know that T_3 is $I_\ell\times I$.

$T_2 \subset T_3$ is proven in a similar way.

~~This also~~ Combined this proves $T_1\neq T_3\neq T_2$