17.2 Q: $A^{\text{closed}} \subseteq Y^{\text{closed}} \subseteq X \Rightarrow A^{\text{closed}} \subseteq X$.

A: Theorem 17.2: $A^{\text{closed}} \Leftrightarrow \exists B^{\text{closed}} \subseteq X$ s.t. $B \cap Y = A$. Theorem 17.1: Closed sets closed under intersections $\Rightarrow A^{\text{closed}} \subseteq X$, since B & Y are closed in X.

(Or give an elementary proof).

17.4 Q: $U^{\text{open}} \subseteq X$, $A^{\text{closed}} \subseteq X$.

Show $U - A$ open $\subseteq X$, $A - U$ open $\subseteq X$.

A: $U - A = U \cap \overbrace{(X - A)}^{\text{open}}$ = ~~union~~ intersection of open sets

$A - U = A \cap \underbrace{(X - U)}_{\text{closed}}$ = intersect of closed sets

17.5 Q: X ordered set w/ order topology.

Show $\overline{(a,b)} \subset [a,b]$.

A: We know $(a,b) \subset [a,b]$. The latter is closed, so 17.6(a) $\Rightarrow \overline{(a,b)} \subset [a,b]$.

(Note $X - [a,b]$ is open & disjoint from (a,b), so we do not really need 17.6(a)).

[We have inequality if $[a,b] - \overline{(a,b)} = \{a,b\} \vee \{a\} \vee \{b\}$.

e.g. if there $\exists$ some c s.t. $[b,c)$ is open]

If X is a linear continuum we have equality.

In fact property (8) p. 30 is equivalent, i.e. $\forall \; x < y \; \exists \; z$ s.t. $x < z < y$.

(It is not equivalent to a linear continuum as $\mathbb{Q}$ satisfies the above, but does not have LUB)

($\{1 < 2 < 3\}$ is another nice example to keep in mind)

17.6(a) Q: $A \subset B \Rightarrow \bar{A} \subset \bar{B}$ (see definition of closure p. 95)

A: $\bar{B}$ is closed & contains $A \Rightarrow \bar{B} \supset \bar{A}$.

(b) Q: $\overline{A \cup B} = \bar{A} \cup \bar{B}$

A: $\bar{A} \cup \bar{B}$ is closed & contains $A \cup B$

$\Rightarrow \overline{A \cup B} \subset \bar{A} \cup \bar{B}$.

$\overline{A \cup B}$ is closed, contains A, & $B \Rightarrow \bar{A} \subset \overline{A \cup B}$

$\bar{B} \subset \overline{A \cup B} \Rightarrow \bar{A} \cup \bar{B} \subset \overline{A \cup B}$.

(c) Q: $\overline{\bigcup A_\alpha} \supset \bigcup \bar{A}_\alpha$. ~~$\bigcup A_\alpha$ is clo~~

A: $\overline{\bigcup A_\alpha}$ is closed & contains $A_\alpha \; \forall \alpha$

$\Rightarrow \overline{\bigcup A_\alpha} \supset \bar{A}_\alpha \; \forall \alpha \Rightarrow \overline{\bigcup A_\alpha} \supset \bigcup_\alpha \bar{A}_\alpha$

Example of strict inequality: $A_n = [1/n, 1]$

Then $\bigcup_{n \in \mathbb{Z}_+} A_n = (0, 1] \subsetneq [0, 1]$.

17.7 Q: Criticize proof...

A: There could be a ~~smaller~~ different $U' \neq U$ s.t.

$A_\alpha \cap U' = \emptyset$, eg in 17.6 (c). any ~~$(a, b)$~~

$(-\varepsilon, \varepsilon)$ intersects some $[1/n, 1]$, nonempty,

but choose $(-\delta, \delta)$ s.t. $(-\delta, \delta) \cap [1/n, 1] = \emptyset$ ($\delta < 1/n$)

17.8

(a) Q: $\overline{A \cap B} \subset \bar{A} \cap \bar{B}$

A: $\bar{A} \supset A$, $\bar{B} \supset B \Rightarrow \bar{A} \cap \bar{B} \supset A \cap B$ (closed)

$\Rightarrow \bar{A} \cap \bar{B} \supset \overline{A \cap B}$. Strict, since $\overline{(0,1) \cap (1,2)}$

$= \emptyset \subset [0,1] \cap [1,2] = [1,1]$

(b) Q: $\overline{\bigcap A_\alpha} \subset \bigcap \bar{A}_\alpha$.

A: Same as ~~above~~ in (a)

(c) Q: $\overline{A - B} \supset \bar{A} - \bar{B}$ — strict since $\overline{[0,1] - \{0,1\}} \supset \emptyset$

A: $x \in \bar{A} - \bar{B} \Rightarrow \exists U_x$ s.t. $U_x \cap B = \emptyset$ & $\forall V_x$: $U_x \cap V \cap A \neq \emptyset$

$\Rightarrow U \cap V \cap (A - B) \neq \emptyset \; \forall V \Rightarrow x \in \overline{A - B}$

3

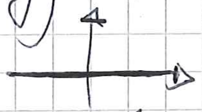
17.13 Q: X Hausdorff $\Leftrightarrow$ $\Delta \subset X \times X$ is closed.

A: Δ closed $\Leftrightarrow$ $\forall\, x \times y$ $x \neq y$ $\exists\, U \times V$
$x \in U^{\text{open}} \subseteq X$, $y \in V^{\text{open}} \subseteq X$ s.t. $U \times V \cap \Delta = \emptyset$
i.e. $U \cap V = \emptyset \Leftrightarrow X$ is Hausdorff.

17.20 Oops, I misread the exercise & only found closure

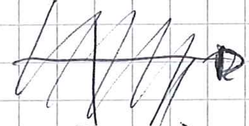
Use: Boundary = Closure − Interior

Q: Find ~~(boundary)~~ closure & interior of: $\subseteq \mathbb{R}^2$

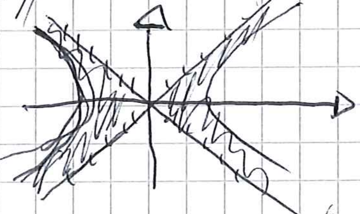
$A = \{x \times y \mid y = 0\}$  $\operatorname{Int} A = \emptyset$, $\bar{A} = A$

$B = \{x \times y \mid x > 0, y \neq 0\}$  $\operatorname{Int} B = B$, $\bar{B} = [0,\infty) \times \mathbb{R}$

$C = A \cup B$ $\operatorname{Int} C = (0,\infty) \times \mathbb{R}$, $\bar{C} = \bar{A} \cup \bar{B}$

$D = \{x \times y \mid x \in \mathbb{Q}\}$  $\operatorname{Int} D = \emptyset$, $\bar{D} = \mathbb{R}^2$

$E = \{x \times y \mid 0 < x^2 - y^2 \leq 1\}$



$\operatorname{Int} E = \{x \times y \mid 0 < x^2 - y^2 < 1\}$

$\bar{E} = \{x \times y \mid 0 \leq x^2 - y^2 \leq 1\}$

$F = \{x \times y \mid x \neq 0 \ \& \ y \leq \frac{1}{x}\}$

$\operatorname{Int} F = \{x \times y \mid x \neq 0 \ \& \ y < \frac{1}{x}\}$

$\bar{F} = $ ~~$\{x \times y \mid x \neq 0 \ \& \ y \leq$~~ $F \cup \{0 \times y \mid y \in \mathbb{R}\}$

18.1 Q: $f: \mathbb{R} \to \mathbb{R}$: Show ε-δ definition $\Rightarrow$ open set definition

A: Must show, $U \overset{\text{open}}{\subseteq} \mathbb{R} \Rightarrow f^{-1}(U) =$ open.
~~Suffices~~ consider $x \in f^{-1}(U)$, for some ε $f(x) \in (f(x)-\varepsilon, f(x)+\varepsilon) \subseteq U$. Choose δ s.t. $\forall\, y \in (x-\delta, x+\delta)$ $|f(y) - f(x)| < \varepsilon$, i.e. $f(y) \in (f(x)-\varepsilon, f(x)+\varepsilon) \subseteq U$
i.e. $(x-\delta, x+\delta) \subseteq f^{-1}(U)$.
Hence, around any $x \in f^{-1}(U)$ $\exists$ open nbhd $\subseteq f^{-1}(U) \Rightarrow f^{-1}(U)$ is open.

18.2 Q: $f: X \to Y$ cont. x limit point of $A \subseteq X$
is $f(x)$ a limit point of $f(A)$?

A: No. Consider the constant function.
Then $f(A) = *$ has no limit points.