Topology problem set.

Recall: Y an (linearly/totally) ordered set means equipped w/ $<$ satisfying
i) $\forall x \neq y$, $x<y$ or $y<x$, ii) $x \nless x$ $\forall x$, iii) transitive.
ii) Order topology has subbase given by rays $(\leftarrow, a)$, $(b, \rightarrow)$
iii) Ordered set w/ order top. is Hausdorff.

Exercise 8: i) $f, g: X \to Y$ cont, Y order top. Show $U = \{x \mid f(x) \leq g(x)\}$ is closed.

We can do this by showing $U^c = \{x \mid f(x) > g(x)\}$ is open.
Pick x ~~st~~ $\in U^c$. We find Nbhd of x in U^c.

Since Y Hausdorff, find opens separating $f(x)$, $g(x)$. WLOG may assume in the base for Y. So

$f(x) \in (a_1, a_2)$ $g(x) \in (b_1, b_2)$.

[We must have $b_2 \leq a_1$ as $f(x) > g(x)$.]

Now consider the open $N(x) = f^{-1}((a_1, a_2)) \cap g^{-1}((b_1, b_2))$.
Clearly $x \in N(x)$, furthermore, $N(x) \subseteq U^c$. Let $y \in N(x)$.
then $f(y) \in (a_1, a_2)$ & $g(y) \in (b_1, b_2)$ So
$f(y) > a_1 \geq b_2 > g(y)$ so $f(y) > g(y)$ and $y \in U^c$.
Hence U^c is open. Why? $U^c = \bigcup_{x \in U^c} N(x)$, and so U is closed.

ii) Show $\min\{f, g\}$ is continuous.
We use pasting lemma. Let $U_f = \{x \mid f(x) \leq g(x)\}$
$U_g = \{x \mid g(x) \leq f(x)\}$. These sets are closed.
Then $f|_{U_f}(x) = \min\{f,g\}(x) = f(x)$ is continuous on U_f as f is cont. Similarly, $g|_{U_g}(x) = \min\{f,g\}(x) = g(x)$ is cont on U_g.
For $x \in U_f \cap U_g$, $f|_{U_f}(x) = f(x) = g(x) = g|_{U_g}(x)$. So
These paste together to give cont. function, namely $\min\{f, g\}$.

Exercise 9: Proof by induction: $\forall$ space X w/ cover $\{U_1, \dots, U_n\}$ $f: X \to Y$, $f|_{U_i}$ is cont. then f is cont.

Proof: i) if $n=1$, this is tautology.
ii) Assume for all ~~spaces X~~ functions $f: X \to Y$ when X is covered by n sets $U_1, \dots, U_n$ and $f|_{U_i}$ is cont, then f is cont.
Now say we are given $f: X \to Y$ and covering $U_1, \dots, U_{n+1}$.
Then ~~$U_1, \dots, (U_n \cup U_{n+1})$~~ $U_1, \dots, U_n$ is a cover of $\bigcup_{i=1}^{n} U_i$
and shows $f|_{\bigcup_{i=1}^n U_i}$ is cont by ind hyp. Pasting lemma gets f cont.

9b) $f:[0,1]\to[0,1]$. $f(x)=\begin{cases} x & \text{if } x>0 \\ 1 & \text{if } x=0.\end{cases}$

$U_n=[\frac{1}{n},1]$, $U_0=\{0\}$

f is not cont. $f^{-1}((\frac{1}{2},1])$ not open.

9c) Say $\{U_\alpha\}$ is loc. fin. cover of X. $f|_{U_\alpha}: U_\alpha\to Y$ is cont.

For any $x\in X$, pick a nbhd $N(x)$ for which $\{U_\alpha \mid U_\alpha\cap N(x)\neq\emptyset\}$ is finite, then $f|_{\bigcup_{i=1} U_{\alpha_i}}$ is continuous by a).

Now use local def of continuity. Let V be an open about $f(x)$. Then $f|_{\bigcup U_{\alpha_i}}^{-1}(V)$ is open in $\bigcup_{i=1}^{n} U_{\alpha_i}$.

So there is some $A\subseteq X$ open s.t.

$x\in f|_{\bigcup U_{\alpha_i}}^{-1}(V)=(\bigcup U_{\alpha_i})\cap A$. Now set ~~tt~~

$B=A\cap N(x)$, it is open about x, ~~as~~ and furthermore $\forall y\in B$, $f(y)\in V$ as $y\in f|_{\bigcup U_{\alpha_i}}^{-1}(V)$.

So $f(B)\subseteq V$. Hence f is continuous.

13) Coincidence theorem:

$$X \supseteq \bar{A} \supseteq A,\quad \bar{A}\overset{g_1}{\underset{g_2}{\rightrightarrows}} Y,\quad A\xrightarrow{f} Y$$

Y is Hausdorff.

Say $g_1(x)=g_2(x)$ for $x\in A$, g_1,g_2 continuous.

Look at set $U=\{x\in\bar{A} \mid g_1(x)=g_2(x)\}$. We show U is closed and contains A, so $U=\bar{A}$.

Look at $U^c=\{x\in\bar{A} \mid g_1(x)\neq g_2(x)\}$.

~~For a~~ if $U^c=\emptyset$, ~~we are done~~ result follows.

If $x\in U^c$, then pick V_1,V_2 disj. about $g_1(x), g_2(x)$.

Look at the open $g_1^{-1}(V_1)\cap g_2^{-1}(V_2)\subseteq U^c$. So U^c is open result follows.

Cool example!

$$\mathbb{R}\supseteq\mathbb{Q},\quad \mathbb{R}\xrightarrow{\exists!} Y,\quad \mathbb{Q}\xrightarrow[\text{cont}]{\forall} Y$$

Hausdorff

e.g. measure 0 set determines a cont function!

2

§19) $\prod_{\alpha\in J} X_\alpha$ has two tops: $\begin{cases} \text{Box top has basis } \prod U_\alpha,\ U_\alpha \text{ open in } X_\alpha \\ \text{Prod top basis } \prod U_\alpha,\ U_\alpha \text{ open in } X_\alpha, \text{ all but fin many are } U_\alpha = X_\alpha. \end{cases}$

Important prop. of prod top is

$$\begin{tikzcd} \prod X_\alpha \arrow[r, "\pi_\alpha"] & X_\alpha \\ Y \arrow[u, "f"] \arrow[ur, "f_\alpha"'] & \end{tikzcd}$$

$\forall$ family of f_α cont, $\exists f$ which is cont.

Any f is cont iff all f_α are cont.

1) Thm 19.2: If each X_α has basis $\mathcal{B}_\alpha$, show $(\alpha\in J)$

$\{\prod B_\alpha \mid B_\alpha \in \mathcal{B}_\alpha\}$ is basis for box top. of $\prod X_\alpha$.

<u>Proof</u>: We show any nonempty U open in $\prod X_\alpha$ contains ~~an elt of the~~ $\forall x\in U$ a member of the basis about x. Say $x\in U\subseteq \prod X_\alpha$ is open. $\exists \prod U_\alpha \subseteq U$, $x\in \prod U_\alpha$.

[Since boxes are base for the top. Each $U_\alpha = \bigcup_{\beta\in J_\alpha} B_{\alpha,\beta}$, $B_{\alpha,\beta}\in\mathcal{B}_\alpha$. ~~Since~~ Now ~~pick~~ have a family of sets $\{J_\alpha \mid \alpha\in J\}$, Use choice to pick $\beta_\alpha \in J_\alpha$ $\forall \alpha\in J$. Then $\prod B_{\alpha,\beta_\alpha} \subseteq \prod U_\alpha \subseteq U$ is an open box

in product top

each $x_\alpha \in U_\alpha$ has $B_\alpha\in\mathcal{B}_\alpha$ s.t. $x_\alpha \subseteq B_\alpha \subseteq U_\alpha$. Then $\prod B_\alpha$ is in the base, $x\in \prod B_\alpha \subseteq \prod U_\alpha \subseteq U$.

$\left[\prod U_\alpha = \bigcup_{\beta\in\prod J_\alpha} \prod B_{\alpha,\beta_\alpha}\right.$, so claim follows $\left.\right]$

in case of product topology, $\prod U_\alpha \subseteq U$ has all but fin many $U_\alpha = X_\alpha$. Proceed as above, but for all but fin many U_α take ~~$U_\alpha =$~~ $J_\alpha = \{1\}$, $B_{\alpha,1} = U_\alpha = X_\alpha$.

in product top: $\forall$ $\tilde U$ open in $\prod X_\alpha$ $\tilde U \cap \prod A_\alpha$ open in prod. $\forall$ $U\subseteq \prod A_\alpha$ open in prod $\exists$ $\tilde U$ with $\tilde U \cap \prod A_\alpha = U$ & $\tilde U$ open in $\prod X_\alpha$

2) If $\forall \alpha\in J$, $A_\alpha \subseteq X_\alpha$ is a subspace, show $\prod A_\alpha$ is subspace of $\prod X_\alpha$ if both are given box resp. prod.

<u>Proof</u>: We do it for box. U_α is open of A_α iff $\exists \tilde U_\alpha$ open in X_α s.t. $U_\alpha = A_\alpha \cap \tilde U_\alpha$. let ~~$\prod U_\alpha$ be open in $\prod A_\alpha$~~ $U\subseteq \prod A_\alpha$ open. ~~U_α~~ consider one

all but fin many $U_\alpha = X_\alpha$

$U = \bigcup_\beta \prod_\alpha U_{\alpha,\beta}$. ~~We show~~ $\prod U_\alpha$ ~~is~~ Then write $U_\alpha = A_\alpha \cap \tilde U_\alpha$.

Then $\prod U_\alpha = \prod A_\alpha \cap \prod \tilde U_\alpha$, and $\prod \tilde U_\alpha$ is open in $\prod X_\alpha$.

So $\bigcup_\beta \prod_\alpha \tilde U_{\alpha,\beta}$ is open in $\prod X_\alpha$ and $\prod A_\alpha \cap \left(\bigcup \prod \tilde U_{\alpha,\beta}\right) = U$.

For each β all but fin many $\tilde{U}_{\alpha,\beta} = X_\alpha$

If we have $\hat{U}$ open in $\prod X_\alpha$, $\hat{U} \cap \prod A_\alpha$; is ~~this~~ open in $\prod A_\alpha$?

We can write $\hat{U} = \bigcup_\beta \prod_\alpha \tilde{U}_{\alpha,\beta}$. Then $\hat{U} \cap \prod A_\alpha = \left(\bigcup \prod \tilde{U}_{\alpha,\beta}\right) \cap \prod A_\alpha$

$$= \bigcup \left(\prod \tilde{U}_{\alpha,\beta} \cap \prod A_\alpha\right)$$

$$= \bigcup \underbrace{\left(\prod \underbrace{(\tilde{U}_{\alpha\beta} \cap A_\alpha)}_{\text{open in } A_\alpha}\right)}_{\text{open in } \prod A_\alpha}.$$

for each β, all but fin many $= A_\alpha$.

Hence open sets in $\prod A_\alpha$ are exactly those which are obtained from $\hat{U} \cap \prod A_\alpha$, $\tilde{U}$ open in $\prod X_\alpha$.

Proof for product topology should be the same, just add in restrictions.

3) A product of Hausdorff spaces is Hausdorff in both box & product top.

Say $x, y \in \prod X_\alpha$, $x \neq y$. So $\exists\, \gamma \in J$, $x_\gamma \neq y_\gamma$.
pick $A_\gamma, B_\gamma \in X_\gamma$ disjoint $x_\gamma \in A_\gamma$, $y_\gamma \in B_\gamma$.

Then $U_\alpha = \begin{cases} X_\alpha & \alpha \neq \gamma \\ A_\gamma & \alpha = \gamma \end{cases}$ $\quad V_\alpha = \begin{cases} X_\alpha & \alpha \neq \gamma \\ B_\gamma & \alpha = \gamma. \end{cases}$

Then $\prod U_\alpha$, $\prod V_\alpha$ are disjoint and open in prod top. Done.

8) $h : \mathbb{R}^\omega \to \mathbb{R}^\omega$, $\pi_i \circ h(x) = a_i x_i + b_i$.

$\pi_i \circ g(x) = \frac{1}{a_i}(x_i - b_i)$. $h \circ g = id = g \circ h$. Cont by checking proj's.

In box top?

i) h is continuous. $\prod (A_i, B_i) \in \mathbb{R}^\omega$ open basis elt,

$$h^{-1}\left(\prod (A_i, B_i)\right) = \{x \mid A_i < a_i x_i + b_i < B_i \quad \forall i\}$$

$$= \left\{x \;\middle|\; \frac{A_i - b_i}{a_i} < x_i < \frac{B_i - b_i}{a_i} \quad \forall i\right\}$$

$$= \prod \left(\frac{A_i - b_i}{a_i}, \frac{B_i - b_i}{a_i}\right) \quad \text{is open}$$

The inverse to h ~~would be~~ is

$\pi_i \circ g(x) = \frac{1}{a_i}(x_i - b_i) = \frac{1}{a_i} x_i - \frac{b_i}{a_i}$ also cont. So also homeo.

③

Axiom of Choice: ~~Given collection of sets~~ X_α, $\alpha \in J$ ~~disjoint~~.

Let $\mathcal{A}$ be a set of disjoint, nonempty sets. $\exists C$ a set $\forall A \in \mathcal{A}$, $A \cap C$ is a singleton and $C \subseteq \bigcup_{A \in \mathcal{A}} A$.

Claim: AC $\Leftrightarrow$ $\forall$ family of nonempty sets $X_\alpha, \alpha \in J$, $\prod_{\alpha \in J} X_\alpha$ is nonempty.

Proof: "$\Rightarrow$" Form $\tilde{X}_\alpha = X_\alpha \times \{\alpha\}$, and the set $\{\tilde{X}_\alpha \mid \alpha \in J\}$. This is set of nonempty, disj. sets,

$\exists C = \{(x_\alpha, \alpha) \mid \alpha \in J, x_\alpha \in X_\alpha\}$.

Define $f: J \to \bigcup X_\alpha$ by $f(\alpha) = x_\alpha$ where $C \cap \tilde{X}_\alpha = \{(x_\alpha, \alpha)\}$.

Then $f \in \prod X_\alpha$, so $\prod X_\alpha \neq \emptyset$.

"$\Leftarrow$" Let $\mathcal{A}$ be given set of disj. nonempty sets. Form $\prod_{A \in \mathcal{A}} A$. ($\mathcal{A} \to \mathcal{A}$, $A \mapsto A$ is index function)

then Nonempty, so $\exists f: \mathcal{A} \to \bigcup_{A \in \mathcal{A}} A$ w/ $f(A) \in A$ $\forall A \in \mathcal{A}$.

Form then $C = \{f(A) \mid A \in \mathcal{A}\}$ satisfies necessary prop. □

$C \cap A$ contains $f(A)$.

If $a \in C \cap A$, then $a = f(B)$ for some $B \in \mathcal{A}$ and $a \in A$. so $f(B) \in A$ and $f(B) \in B$, so $B = A$ and $a = f(A)$. So $C \cap A = \{f(A)\}$.

□