

Topology problem set.

(1)

Recall: Y an ordered set means equipped w/ $<$ satisfying
 i) $\forall x, y, x < y$ or $y < x$, ii) $x < x$ $\forall x$, iii) transitive.
 ii) Order topology has subbase given by rays $(-\infty, a)$, (b, ∞)
 iii) Ordered set w/ order top. is Hausdorff.

Exercise 8: $f, g: X \rightarrow Y$ cont, Y order top. Show $U = \{x \mid f(x) \leq g(x)\}$ is closed.

We can do this by showing $U^c = \{x \mid f(x) > g(x)\}$ is open.
 Pick $x \in U^c$. We find Nbd of x in U^c .

Since Y Hausdorff, find opens separating $f(x), g(x)$. WLOG may assume in the base for Y . So

$$f(x) \in (a_1, a_2) \quad g(x) \in (b_1, b_2).$$

We must have $b_2 \leq a_1$ as $f(x) > g(x)$.

Now consider the open $N(x) = f^{-1}((a_1, a_2)) \cap g^{-1}((b_1, b_2))$.

Clearly $x \in N(x)$, furthermore, $N(x) \subseteq U^c$. Let $y \in N(x)$,

then $f(y) \in (a_1, a_2)$ & $g(y) \in (b_1, b_2)$ so

$$f(y) > a_1 \geq b_2 > g(y) \quad \text{so } f(y) > g(y) \text{ and } y \in U^c.$$

Hence U^c is open. Why? $U^c = \bigcup_{x \in U^c} N(x)$, and so U is closed.

ii) Show $\min\{f, g\}$ is continuous.

We use pasting lemma. Let $U_f = \{x \mid f(x) \leq g(x)\}$

$U_g = \{x \mid g(x) \leq f(x)\}$. These sets are closed.

Then $f|_{U_f}(x) = \min\{f, g\}(x) = f(x)$ is continuous on U_f as f is

cont. Similarly, $g|_{U_g}(x) = \min\{f, g\}(x) = g(x)$ is cont on U_g .

For $x \in U_f \cap U_g$, $f|_{U_f}(x) = f(x) = g(x) = g|_{U_g}(x)$. So

These paste together to give cont. function, namely $\min\{f, g\}$.

Exercise 9: Proof by induction: \forall space X w/ cover $\{U_1, \dots, U_n\}$
 $f: X \rightarrow Y$, $f|_{U_i}$ is cont. then f is cont.

Proof: i) if $n=1$, this is tautology.

ii) Assume for all spaces X functions $f: X \rightarrow Y$ when X is covered by n sets U_1, \dots, U_n and $f|_{U_i}$ is cont, then f is cont.

Now say we are given $f: X \rightarrow Y$ and covering U_1, \dots, U_n .

Then (U_1, \dots, U_n) is a cover of $\bigcup_{i=1}^n U_i$

and shows $f|_{\bigcup_{i=1}^n U_i}$ is cont by ind hyp. Pasting lemma gets f cont.

9b) $f: [0, 1] \rightarrow [0, 1]$. $f(x) = \begin{cases} x & \text{if } x > 0 \\ 1 & \text{if } x = 0 \end{cases}$

$U_n = [\frac{1}{n}, 1]$, $U_0 = \{0\}$

f is not cont. $f^{-1}((\frac{1}{2}, 1])$ not open,

9c) Say $\{U_\alpha\}$ is loc. fin. cover of X . $f|_{U_\alpha}: U_\alpha \rightarrow Y$ is cont.
 For any $x \in X$, pick a nbhd $N(x)$ for which $\{U_\alpha \mid U_\alpha \cap N(x) \neq \emptyset\}$
 is finite, then $f|_{\cup_{i=1}^n U_{\alpha_i}}$ is continuous by a).

Now use local def of continuity. Let V be an open about

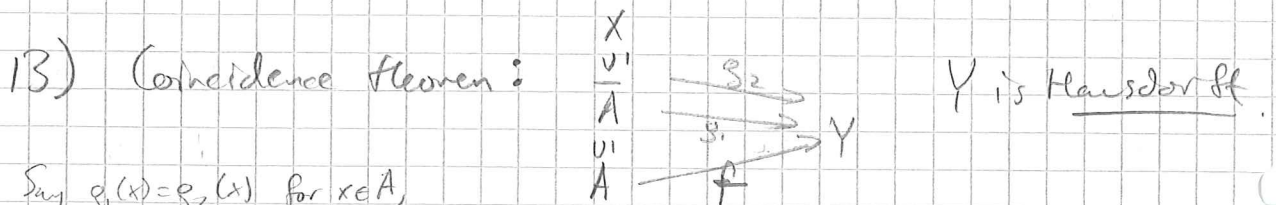
$f(x)$. Then $f|_{\cup_{i=1}^n U_{\alpha_i}}^{-1}(V)$ is open in $\cup_{i=1}^n U_{\alpha_i}$.

So there is some $A \subseteq X$ open s.t.

$x \in f|_{\cup_{i=1}^n U_{\alpha_i}}^{-1}(V) = (\cup U_{\alpha_i}) \cap A$. Now set U

$B = A \cap N(x)$, it is open about x , ~~is~~ and
 furthermore $\forall y \in B, f(y) \in V$ as $y \in f|_{\cup_{i=1}^n U_{\alpha_i}}^{-1}(V)$.

So $f(B) \subseteq V$. Hence f is continuous.



Say $g_1(x) = g_2(x)$ for $x \in A$,
 g_1, g_2 continuous.

Look at set $U = \{x \in \bar{A} \mid g_1(x) = g_2(x)\}$. We show U is closed
 and contains A , so $U = \bar{A}$.

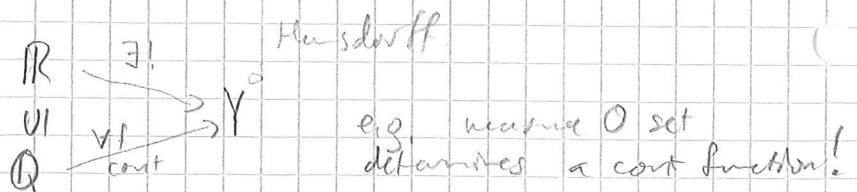
Look at $U^c = \{x \in \bar{A} \mid g_1(x) \neq g_2(x)\}$.

For ea if $U^c = \emptyset$, we are done result follows.

If $x \in U^c$, then pick V_1, V_2 disj. about $g_1(x), g_2(x)$.

Look at the open $g_1^{-1}(V_1) \cap g_2^{-1}(V_2) \subseteq U^c$. So U^c is open
 result follows.

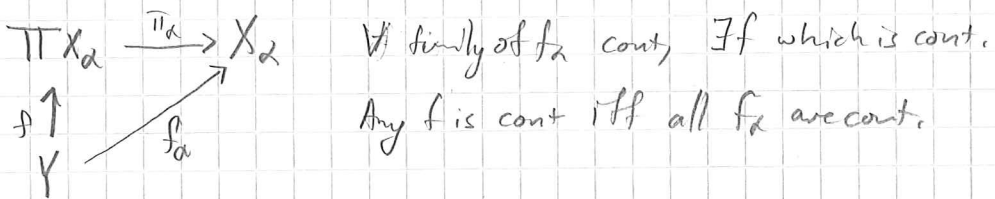
Cool example:



②

§19) $\prod_{\alpha \in J} X_\alpha$ has two tops: $\left\{ \begin{array}{l} \text{Box top basis } \prod U_\alpha, U_\alpha \text{ open in } X_\alpha \\ \text{Prod top basis } \prod U_\alpha, U_\alpha \text{ open in } X_\alpha, \text{ all but fin many are } U_\alpha = X_\alpha \end{array} \right.$

Important prop. of prod top is



1) Thm 19.2: If each X_α has basis \mathcal{B}_α , show $(\alpha \in J)$

$\{\prod B_\alpha \mid B_\alpha \in \mathcal{B}_\alpha\}$ is basis for box top. of $\prod X_\alpha$.

Proof: We show any U open in $\prod X_\alpha$ contains a ~~lot of~~ a number of the basis. ^{nonempty U} ^{$x \in U$} Say $U \subseteq \prod X_\alpha$ is open, $\exists \prod U_\alpha \subseteq U, x \in \prod U_\alpha$.

Since boxes are base for the top. Each $U_\alpha = \bigcup_{\beta \in J_\alpha} B_{\alpha, \beta}, B_{\alpha, \beta} \in \mathcal{B}_\alpha$.

Now pick a family of sets $\{J_\alpha \mid \alpha \in J\}$, use

choice to pick $\beta_\alpha \in J_\alpha \forall \alpha \in J$. Then $\prod B_{\alpha, \beta_\alpha} \subseteq \prod U_\alpha \subseteq U$

is an open box

each $x_\alpha \in U_\alpha$ has $B_\alpha \in \mathcal{B}_\alpha$ s.t. $x_\alpha \in B_\alpha \subseteq U_\alpha$. Then

$\prod B_\alpha$ is in the base, $x \in \prod B_\alpha \subseteq \prod U_\alpha \subseteq U$.

$\prod U_\alpha = \bigcup_{\beta \in \prod J_\alpha} \prod B_{\alpha, \beta_\alpha}$, so claim follows

in case of product topology, $\prod U_\alpha \subseteq U$ has all but fin many

$U_\alpha = X_\alpha$. Proceed as above, but for all but fin many U_α

take $J_\alpha = \{\emptyset\}, B_{\alpha, \emptyset} = U_\alpha = X_\alpha$.

2) If $\forall \alpha \in J, A_\alpha \subseteq X_\alpha$ is a subspace, show $\prod A_\alpha$ is subspace of $\prod X_\alpha$

if both are given box resp. prod.

Proof: We do it for box. U_α is open of A_α iff $\exists \tilde{U}_\alpha$ s.t., \tilde{U}_α open in X_α .

$U_\alpha = A_\alpha \cap \tilde{U}_\alpha$. let $\prod U_\alpha$ be open in $\prod A_\alpha$ $U \subseteq \prod A_\alpha$ open.

$U = \bigcup_{\beta} \prod U_{\alpha, \beta}$. ~~We show~~ $\prod U_\alpha$ is then write $U_\alpha = A_\alpha \cap \tilde{U}_\alpha$.

Then $\prod U_\alpha = \prod A_\alpha \cap \prod \tilde{U}_\alpha$, and $\prod \tilde{U}_\alpha$ is open in $\prod X_\alpha$.

So $\bigcup_{\beta} \prod U_{\alpha, \beta}$ is open in $\prod X_\alpha$ and $\prod X_\alpha \cap (\bigcup_{\beta} \prod \tilde{U}_{\alpha, \beta}) = U$.

in product top:
 \tilde{U} open in $\prod X_\alpha$
 $\tilde{U} \cap \prod A_\alpha$ open in prod.
 $U \subseteq \prod A_\alpha$ open in prod
 \tilde{U} with $\tilde{U} \cap \prod A_\alpha = U$
 \tilde{U} open in $\prod X_\alpha$

all but fin many $U_\alpha = X_\alpha$

If we have \tilde{U} open in $\prod X_\alpha$, $\tilde{U} \cap \prod A_\alpha$; is this open in $\prod A_\alpha$?

We can write $\tilde{U} = \bigcup_{\beta} \prod_{\alpha} \tilde{U}_{\alpha, \beta}$. Then $\tilde{U} \cap \prod A_\alpha = (\bigcup_{\beta} \prod_{\alpha} \tilde{U}_{\alpha, \beta}) \cap \prod A_\alpha$
 $= \bigcup_{\beta} (\prod_{\alpha} \tilde{U}_{\alpha, \beta} \cap \prod A_\alpha)$
 $= \bigcup_{\beta} (\prod_{\alpha} (\tilde{U}_{\alpha, \beta} \cap A_\alpha))$

Hence open sets in $\prod A_\alpha$ are exactly those which are obtained from $\tilde{U} \cap \prod A_\alpha$, \tilde{U} open in $\prod X_\alpha$.

open in A_α
 \uparrow
 open in $\prod A_\alpha$
 for each β , all but fin many = A_α .

Proof for product topology should be the same, just add in restrictions.

3) A product of Hausdorff spaces is Hausdorff in both box & product top.

say $x, y \in \prod X_\alpha$, $x \neq y$. So $\exists \alpha \in J$, $x_\alpha \neq y_\alpha$.

pick $A_\alpha, B_\alpha \in X_\alpha$ disjoint, $x_\alpha \in A_\alpha$, $y_\alpha \in B_\alpha$.

Then $U_\alpha = \begin{cases} X_\alpha & \alpha \neq \alpha \\ A_\alpha & \alpha = \alpha \end{cases}$ $V_\alpha = \begin{cases} X_\alpha & \alpha \neq \alpha \\ B_\alpha & \alpha = \alpha \end{cases}$

Then $\prod U_\alpha, \prod V_\alpha$ are disjoint and open in prod top. Done.

8) $h: \mathbb{R}^n \rightarrow \mathbb{R}^n$, $\pi_i \circ h = a_i x_i + b_i$.

$\pi_i \circ g(x) = \frac{1}{a_i} (x_i - b_i)$. $h \circ g = id = g \circ h$. Cont by checking proj's.

In box top?

i) h is continuous. $\prod (A_i, B_i) \in \mathbb{R}^n$ open, basis elt.

$h^{-1}(\prod (A_i, B_i)) = \{x \mid A_i < a_i x_i + b_i < B_i \ \forall i\}$
 $= \{x \mid \frac{A_i - b_i}{a_i} < x_i < \frac{B_i - b_i}{a_i} \ \forall i\}$
 $= \prod (\frac{A_i - b_i}{a_i}, \frac{B_i - b_i}{a_i})$ is open

The inverse to h would be is

$\pi_i \circ g(x) = \frac{1}{a_i} (x_i - b_i) = \frac{1}{a_i} x_i - \frac{b_i}{a_i}$ also conti. So

also homeo.

For each β
 all but fin many
 $\tilde{U}_{\alpha, \beta} = X_\alpha$

③

Axiom of Choice: Given collection of sets $X_\alpha, \alpha \in J$ disjoint.

Let \mathcal{A} be a set of disjoint, ^{nonempty} sets. $\exists C$ a set $\forall A \in \mathcal{A}$,

$A \cap C$ is a singleton and $C \subseteq \bigcup_{A \in \mathcal{A}} A$.

Claim: $AC \Leftrightarrow \forall$ family of sets $X_\alpha, \alpha \in J$, $\prod_{\alpha \in J} X_\alpha$ is nonempty.

Proof: " \Rightarrow " Form $\tilde{X}_\alpha = X_\alpha \times \{\alpha\}$, and the set

of $\{\tilde{X}_\alpha \mid \alpha \in J\}$. This is a set of nonempty, disjoint sets.

$\exists C = \{(x_\alpha, \alpha) \mid \alpha \in J\}$.

Define $f: J \rightarrow \bigcup X_\alpha$ by $f(\alpha) = x_\alpha$ where $C \cap \tilde{X}_\alpha = \{(x_\alpha, \alpha)\}$.

Then $f \in \prod X_\alpha$, so $\prod X_\alpha \neq \emptyset$.

" \Leftarrow " Let \mathcal{A} be given. Form $\prod_{A \in \mathcal{A}} A$. ($\mathcal{A} \rightarrow \mathcal{A}$ is index function)

Then nonempty, so $\exists f: \mathcal{A} \rightarrow \bigcup_{A \in \mathcal{A}} A$ w/ $f(A) \in A \forall A \in \mathcal{A}$.

Form then $C = \{f(A) \mid A \in \mathcal{A}\}$ satisfies necessary prop. \square

$C \cap A$ contains $f(A)$. ^{see B set.}

If $a \in C \cap A$, then $a = f(B)$ and $a \in A$. so $f(B) \in A$

and $f(B) \in B$, so $B = A$ and $a = f(A)$. so $C \cap A = \{f(A)\}$. \square

