1 30.09 MAT4500

20.1 (a) Q: $d'(x,y) = \sum |x_i - y_i|$, show d' is a metric & induces usual topology on $\mathbb{R}^n$.

A: (1) $d'(x,y) \geq 0 \quad \forall x,y$, since ... $\Big\}$ Obvious
(2) $d'(x,y) = d'(x,y)$
(3) Triangle inequality from triang ineq of $|\cdot|$.

Note: $\sum (x_i - y_i)^2 \leq \left(\sum |x_i - y_i|\right)^2 \leq \left(n \cdot \max_i |x_i - y_i|\right)^2$

$\leq n^2 \sum (x_i - y_i)^2$. Hence, $d(x,y) \leq d'(x,y)$

$\leq n\, d(x,y)$, so $B_d(x,\varepsilon) \supseteq B_{d'}(x,\varepsilon) \supseteq B_d(x, \frac{1}{n}\varepsilon)$

$\forall\ \varepsilon, x$, so d & d' induce the same toplogy on $\mathbb{R}^n$.

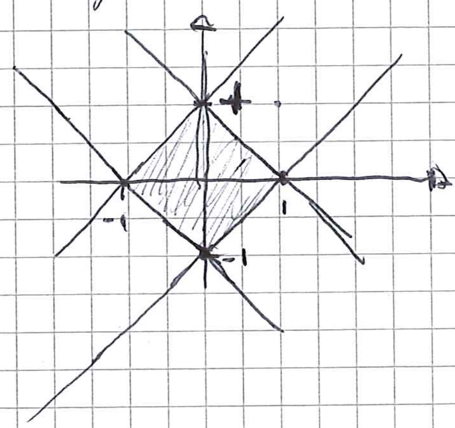
Basis element for d': consider $B_{d'}(0,1)$, i.e.

points $y = (y_1, y_2)$ s.t. $|y_1| + |y_2| < 1$

$\iff y_1 + y_2 < 1$ & $y_1 - y_2 < 1$ & $-y_1 + y_2 < 1$

& $-y_1 - y_2 < 1$

These four equations each describe a shifted and rotated halfplane. If we draw the halfplanes we get:



20.1(b) Q: Show $d''(x,y) = \left(\sum |x_i - y_i|^p\right)^{1/p}$ induces the usual topology on $\mathbb{R}^n$.

A: $\sum |x_i - y_i|^p \le \left(\sum |x_i - y_i|\right)^p$

$\le (n \cdot \max |x_i - y_i|)^p \le n^p \cdot \max |x_i - y_i|^p$

$\le n^p \cdot \sum |x_i - y_i|^p$

$\Rightarrow \quad d''(x,y) \le d'(x,y) \le n\, d''(x,y)$

Argue as in 20.1(a).

26.3(a) Q: (X,d) metric spa, show $d: X \times X \to \mathbb{R}$ is cont.

A: We check ε-δ-continuity. Given $\varepsilon > 0$ & $x_0 \times y_0 \in X \times X$, then $|d(x_0, y_0) - d(x,y)|$

$\le d(x_0, x) + d(y, y_0)$ = metric on $X \times X$, so

(Variant of reverse triang ineq.) pick $\delta = \varepsilon$.

(Note, a metric on $X \times X$ is given by $d'((x,y),(z,w)) = d(x,z) + d(y,w)$. I leave it to you to check that this induces the usual product topology on $X \times X$).

(b) Q: Let X' have the same underlying set as X & assume $d: X' \times X' \to \mathbb{R}$ is cont. Show tplgy on X' is finer than tplgy on X.

A: Consider $B_d(x_0, \varepsilon)$ a basis element of X.

Note that $f: X' \to \{x_0\} \times X' \subseteq X' \times X' \xrightarrow{d} \mathbb{R}$ is cont.

$x \mapsto x_0 \times x$

So $f^{-1}((-\varepsilon, \varepsilon)) = B_d(x_0, \varepsilon)$ is open in X', hence X' is finer than X.

20.4 (a) Q: Consider product, uniform & box tplgy on $\mathbb{R}^\omega$. In which tplgies are

$$f(t) = (t, 2t, 3t, \ldots)$$
$$g(t) = (t, t, \ldots)$$
$$h(t) = (t, \tfrac{1}{2}t, \ldots) \qquad \text{cont?}$$

A: Note: Theorem 20.4 $\Rightarrow$ $\mathcal{T}_{prod} \subseteq \mathcal{T}_{uni} \subseteq \mathcal{T}_{Box}$ and if ~~ℓ is cont~~ a function ℓ is cont in $\mathcal{T}' \supseteq \mathcal{T}$ (tplgies on $\mathbb{R}^\omega$) then ℓ is cont. in $\mathcal{T}$. Similarly if ℓ is discont. in $\mathcal{T}$ it is discont in $\mathcal{T}'$.

By Theorem 19.6 all the functions are cont. in $\mathcal{T}_{prod}$.

Since $f^{-1}(B_{\bar\rho}((0,0,\ldots),1)) = \{0\}$ f is discont in the uniform & box topology.

With ε-δ continuity we see that g & h are cont. in uniform topolgy. Indeed, given $t \in \mathbb{R}$ $1 > \varepsilon > 0$, then $\bar\rho(g(t), g(s)) < \varepsilon$ for all s s.t. $|t - s| < \varepsilon$. Similarly for h.

(If you do not want to use ε-δ-continuity, you can use 20.6 (c))

g & h are discont in the box topology.

E.g. let $u = \prod\left(-\frac{1}{n^2}, \frac{1}{n^2}\right)$, then

$$g^{-1}(u) = \{0\} = h^{-1}(u).$$

$(0 < \varepsilon < 1)$

20.6 (a) Q: Let $U(x^{(x_1, x_2, \ldots)}, \varepsilon) = (x_1 - \varepsilon, x_1 + \varepsilon) \times \ldots \times (x_n - \varepsilon, x_n + \varepsilon) \times \ldots$

Show $U(x, \varepsilon) \neq B_{\bar\rho}(x, \varepsilon)$.

A: Let $y = (x_1, x_1 + \varepsilon(1 - \tfrac{1}{2}), \ldots, x_1 + \varepsilon(1 - \tfrac{1}{n}), \ldots) \in U(x, \varepsilon)$, but $y \notin B_{\bar\rho}(x, \varepsilon)$.

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20.6 (b) Q: Show $\mathcal{U}(x,\varepsilon)$ is not open.

A: There is no open ball around y contained in $B_{\bar{\rho}}(x,\varepsilon) \setminus \mathcal{U}(x,\varepsilon)$

(c) Q: Show $B_{\bar{\rho}}(x,\varepsilon) = \bigcup_{\delta<\varepsilon} \mathcal{U}(x,\delta)$

(Assume $\varepsilon<1$)

A: $y \in \mathcal{U}(x,\delta) \Rightarrow |x_i - y_i| < \delta < \varepsilon$ $\forall i$ $\Rightarrow \sup |x_i - y_i| \leq \delta < \varepsilon$ $\Rightarrow y \in B_{\bar{\rho}}(x,\varepsilon)$.

$y \in B_{\bar{\rho}}(x,\varepsilon) \Rightarrow \sup |x_i - y_i| =: \delta < \varepsilon$

$\Rightarrow y \in \mathcal{U}\left(x, \frac{\delta+\varepsilon}{2}\right)$

21.1 Q: $A \subset X$, d metric on X, show $d|_{A\times A}$ is a metric for subspace top on X

A: I leave it to you to check that $d|_{A\times A}$ is a metric.

$U \subseteq A$ open $\Leftrightarrow$ $U = V \cap A$, V open in X, may write V as

$$V = \bigcup_{x,\varepsilon} B_d(x,\varepsilon) \Rightarrow U = V \cap A =$$

$$\bigcup_{x,\varepsilon} B_d(x,\varepsilon) \cap A = \bigcup_{x,\varepsilon} B_{d|_{A\times A}}(x,\varepsilon)$$

$\Rightarrow$ Subspace top on $A \subseteq$ metric top. on A

The reverse inclusion is clear, since $B_d(x,\varepsilon) \cap A = B_{d|_{A\times A}}(x,\varepsilon)$.

21.8 Q: X top spc, Y metric spc, $\{f_n : X \to Y\}_{n \in \mathbb{Z}_+}$, $\{x_n\}_{n \in \mathbb{Z}}$ converging to x. ~~$\lim x_n = x$~~

Show: If f_n converges uniformly to f then $f_n(x_n)$ converges to $f(x)$.

A: $d(f(x), f_n(x_n)) \leq d(f(x), f(x_n))$

$+ d(f(x_n), f_n(x_n)) < \varepsilon$

Pick N s.t. $d(f(x), f(x_n)) < \varepsilon/2$ by continuity of f (See Theorem 21.6) $\forall\, n \geq N$

& s.t. $d(f(x_n), f_n(x_n)) < \varepsilon/2$ $\forall\, n \geq N$ by uniform convergence of f_n to f.

21.9 $f_n : \mathbb{R} \to \mathbb{R}$

(a) Q: $f_n(x) := \dfrac{1}{n^3 (x - 1/n)^2 + 1}$

$f(x) := 0 \qquad f : \mathbb{R} \to \mathbb{R}$.

Show $f_n(x) \to f(x)$ $\forall\, x$.

A: $\displaystyle\lim_{n \to \infty} \frac{1}{n^3 (x - 1/n)^2 + 1} = 0 = f(x)$, since ...

(b) Q: Show f_n does not converge uniformly to f.

A: maximum of f_n is $(1/n, 1)$.

hence $\sup_x |f_n(x) - f(x)| = 1 \quad \forall\, n$.

I.e. we do not have uniform convergence

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21.10

Q: Show the following sets are closed in $\mathbb{R}^2$

A: $A = \{x \times y \mid xy = 1\} = f^{-1}(\{1\})$, where $f(x,y) = xy$.

$S^1 = \{x \times y \mid x^2 + y^2 = 1\} = g^{-1}(\{1\})$, where $g(x,y) = x^2 + y^2$

$B^2 = \{x \times y \mid x^2 + y^2 \leq 1\} = g^{-1}([0,1])$.

Note that polynomials are continuous, and the inverse image of a closed set is closed.

20.4

(b) All the sequences converge to (only) $0 = (0, 0, \ldots)$ in the product topology. Hence, we check if the sequences converge to 0 in other tplgies.

We have $\bar{\rho}(0, w_n) = n$

$\bar{\rho}(0, x_n) = \frac{1}{n}$

$\bar{\rho}(0, y_n) = \frac{1}{n}$

$\bar{\rho}(0, z_n) = \frac{1}{n}$,

so x_n, y_n & z_n converges in the uniform topology.

Let $U = \prod_{n \in \mathbb{Z}_+} (-1, \frac{1}{n})$. Then $x_n \notin U \ \forall n$

$y_n \notin U \ \forall n$, so x_n & y_n do not converge in the box topology.

z_n converge in the box topology.

21.2

A: f is injective, since

$f(x_1) = f(x_2) \Leftrightarrow d_Y(f(x_1), f(x_2)) = 0$

$\Leftrightarrow d_X(x_1, x_2) = 0$

$\Leftrightarrow x_1 = x_2$

f is a homeomorphism onto its image. Indeed, define

$g: f(X) \to X$, by $g = f^{-1}$, i.e.

$g(y) = x$, where x is the unique x s.t. $f(x) = y$.

Then g is continuous, since

$d_X(g(y_1), g(y_2)) = d_Y(f(g(y_1)), f(g(y_2)))$

$= d_Y(y_1, y_2)$, & use ε-δ - continuity.