22.1 Q: Let $p: \mathbb{R} \to \{a, b, c\} =: A$, be defined by $p(x) = \begin{cases} a & x>0 \\ b & x<0 \\ c & x=0 \end{cases}$

check that the quotient topology induced on A is: $\{\emptyset, A, \{a\}, \{b\}, \{a,b\}\}$.

A: ~~clearly~~ since $p^{-1}(\{a\}) = (0,\infty)$, $p^{-1}(\{b\}) = (-\infty, 0)$ & $p^{-1}(\{a,b\}) = \mathbb{R} - 0$, are all open sets, $\{a\}$, $\{b\}$ & $\{a,b\}$ are all open.

We check that there are no ~~further~~ more open sets, i.e. $\{a,c\}$, $\{b,c\}$ & $\{c\}$ are not open. Indeed, this is true, since $p^{-1}(\{a,c\}) = [0,\infty)$ $p^{-1}(\{b,c\}) = (-\infty, 0]$ & $p^{-1}(c) = \{0\}$, none of which are open.

22.2 (a) Q: $p: X \to Y$ cont. Show that if $\exists$ $f: Y \to X$ cont. s.t. $pf = id_Y$ ~~then~~ p is a quotient map.

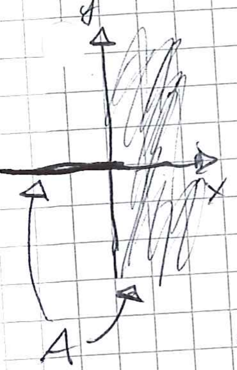
A: p is surjective. Indeed, given $y \in Y$ $p(f(y)) = y$.

$U \subset Y$ is open iff $p^{-1}(U)$ is open. Indeed, if U is open, then $p^{-1}(U)$ is open since p is continuous. If $p^{-1}(U)$ is open, then $f^{-1}p^{-1}(U) = (pf)^{-1}(U) = id_Y^{-1}(U) = U$ is open since f is continuous.

(b) Q: Given $A \subset X$ & $\underbrace{r: X \to A \text{ cont}}_{\text{a retraction}}$ s.th. (i.e. $r(a) = a$ $\forall a \in A$. Show r is a quotient map.

A: Apply (a) to r & $i: A \to X$, the inclusion map.

22.3 Q: Let $\pi_1 : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ ($x\times y \mapsto x$), $A := \{x\times y \in \mathbb{R}\times\mathbb{R} \mid x \geq 0 \text{ or } y = 0\}$, let $q : A \to \mathbb{R}$ be $q = \pi_1|_A$.



Show q is a quotient map.
Show q is neither open nor closed.

A: q is surjective, since given $x \in \mathbb{R}$ $q(x\times 0) = x$ ($x\times 0 \in A$)
q is cont, it is the restriction of a cont. map.
If $U \subseteq \mathbb{R}$ & $q^{-1}(U)$ is open then U is open.
Indeed, if $x \in U$, then $q^{-1}(\{x\}) = \begin{cases} \{x\times 0\} & x<0 \\ \{x\times y \mid y\in\mathbb{R}\} & x \geq 0 \end{cases}$
since $q^{-1}(U)$ is open, there exists a ball s.t. $B(x\times 0, \varepsilon) \cap A \subseteq q^{-1}(U)$.
Then $q(B(x\times 0, \varepsilon)\cap A) = B(x,\varepsilon) \subseteq U$,
So U is open.
q is not open, since $q(B(0\times 1, 1)\cap A) = [0,1)$
q is not closed, since $q(\{x\times y \mid xy = 1\}\cap A) = (0,\infty)$.

22.4 (a) Q: Consider ~~Define~~ the equivalence relation $x_0\times y_0 \sim x_1 \times y_1$ iff $x_0 + y_0^2 = x_1 + y_1^2$ on $\mathbb{R}^2 = X$
Show that the quotient space X^* is homeo to $\mathbb{R}$.

A: Define $g : \mathbb{R}^2 \to \mathbb{R}$, by $x\times y \mapsto x + y^2$.
Then g is cont, surj. & a quotient map.
Indeed, ~~g is an open map~~ given $U \subseteq \mathbb{R}$ s.t. $g^{-1}(U)$ is open. consider $x \in U$ and a ball around $x\times 0$ in $g^{-1}(U)$.

Assume $(x+y)\times 0$ $(x-y)\times 0$ is in that nbhd.
Then $(x-y, x+y) \subseteq U$, so U is open.
Using Corollary 22.3 g induces a homeomorphism from X^* to $\mathbb{R}$.

(b) Q: consider the equivalence relation $x_0\times y_0 \sim x_1\times y_1$ iff $x_0^2+y_0^2 = x_1^2+y_1^2$. Show that the quotient space is homeomorphic to $[0,\infty)$.

A: Define $g : \mathbb{R}^2 \to [0,\infty)$ by $x\times y \mapsto x^2+y^2$.
Then g is cont, surj & a quotient map.
Indeed, given $U \subseteq [0,\infty)$ s.t. $g^{-1}(U)$ is open, consider $x \in U$ & a ball ~~around~~ $B(\sqrt{x}\times 0, \varepsilon)$. If $x>0$, pick $\varepsilon < \sqrt{x}$: $\sqrt{y}\times 0 \in B(\sqrt{x}\times 0, \varepsilon) \subseteq g^{-1}(U)$. Then $\forall\, y \in (x-\varepsilon^2, x+\varepsilon)$, so $(x-\varepsilon^2, x+\varepsilon) \subseteq U$ (so U is open). If $x=0$, pick $\varepsilon<1$, then $[0,\varepsilon) \subseteq U$.

22.5 Q: If $p : X \to Y$ is open, $A \subseteq X$ open, show $q : A \to p(A)$ is an open map (defined by restricting p).

A: Let $U \subseteq A$ be open. Then U is open in X, so $q(U) = p(U)$ is open in Y, and contained in Y, so q is open.

23.9 Q: $A \subsetneq X$, $B \subsetneq Y$. Show $X\times Y - A\times B$ is connected.

A: Pick $x_0 \in X-A$, $y_0 \in Y-B$.
Write $X\times Y - A\times B = \bigcup_{x\in X-A}(\{x\}\times Y \cup X\times\{y_0\}) \cup \bigcup_{y\in Y-A}(X\times\{y\}\cup \ldots$

The point $x_0\times y_0$ is common to all the spaces.
Apply Theorem 23.3.

23.11 Q: $p: X \to Y$ is a quotient map.
Assume $p^{-1}(\{y\})$ is conn. $\forall y \in Y$, & Y is conn.
Show X is conn.

A: Assume X is disconnected. I.e. $\exists U, V \neq \emptyset$ s.t. $U \cap V = \emptyset$ $U \cup V = X$. Since $p^{-1}(\{y\})$ is conn. $\forall y \in Y$, we have $p^{-1}(\{y\}) \subseteq U$

(cf. Lemma 23.2) or $p^{-1}(\{y\}) \subseteq V$ $\forall y$. Then $p(U) \cap p(V) = \emptyset$ & $p^{-1}(p(U)) = U$ & $p^{-1}(p(V)) = V$.
Hence $p(U)$ & $p(V)$ form a separation of Y.
This contradicts Y being connected.

23.12 Q: $Y \subset X$. Let X & Y be conn. Assume A & B form a separation of $X - Y$.
Show $Y \cup A$ & $Y \cup B$ are connected.

A: Assume $Y \cup A$ is disconn. i.e. $\exists U, V \subseteq Y \cup A$ open s.t. $U \cup V = Y \cup A$, $U \cap V = \emptyset$.
may assume $Y \subseteq U$, since Y is conn.
~~Want to show $U \cup B$ is open & closed.~~
~~Assume~~ Write $U = U' \cap (Y \cup A)$ U' open in X
$V = V' \cap (Y \cup A)$ V' open in X
& $A = A' - Y$, $B = B' - Y$, A', B' open in X.
Then $U = U' \cap (Y \cup A) = U' \cap (Y \cup (A' - Y)) = U' \cap A'$ = open in X, and similarly for V. Furthermore $U \cup B = U \cup Y \cup (B' - Y) = U \cup B'$ is open in X.
We have $(U \cup B) \cap V = \emptyset$ & $U \cup V \cup B = X$.
So $U \cup B$ & V form a separation of X. This contradicts X being connected. The proof for $Y \cup B$ being connected is almost identical.

23.12 Q: $Y \subset X$. Let X & Y be conn.
Assume A & B form a separation of $X - Y$. Show $Y \cup A$ & $Y \cup B$ are conn.

A: Assume $Y \cup A$ is disconn. i.e.
$\exists$ U, V open (closed) $\subseteq Y \cup A$, s.t. $U \cup V = Y \cup A$, $U \cap V = \emptyset$. May assume $Y \subseteq U$ since Y is conn.

Note: $\overline{Y \cup A} = Y \cup A$. Indeed, $X = Y \cup A \cup B$, since A is closed in $X - Y$, $\exists$ C closed $\subseteq X$ s.t. $A = (X - Y) \cap C$, i.e. $C \subset \cancel{X} \cup A$
$\Rightarrow$ $A \subset C \subset Y \cup A \Rightarrow \bar{A} \subset Y \cup A$.
Note: $\bar{A} \cap B = \emptyset$.
Let $U' \subseteq X$, be a closed set s.t. $U' \cap (Y \cup A) = U$. Then $\bar{A} - U'$, $U' \cup B$ form a separation of X. Indeed, $\bar{A} - U'$ closed $\subseteq$

$\bar{A} - U'$ closed: { $\bar{A} \overset{\text{closed}}{\subseteq} X$, so $\bar{A} - U'$ is closed in X.

$U' \cup B$ closed: { As above, $\overline{Y \cup B} = Y \cup B$. Since $Y \subseteq U'$ we get $\overline{U' \cup B} = \overline{U'} \cup \bar{B} \subseteq U' \cup Y \cup B = U' \cup B$, so $U' \cup B$ is closed.

$(\bar{A} - U') \cap (U' \cup B) = \emptyset$ { We have $(\bar{A} - U') \cap (U' \cup B) = (\bar{A} - U') \cap U' \cup ((\bar{A} - U') \cap B) \subseteq \emptyset \cup (A \cap B) = \emptyset \cup \emptyset = \emptyset$.

$(\bar{A} - U') \cup (U' \cup B) = X$ { We have $(\bar{A} - U') \cup U' \cup B = \bar{A} \cup U' \cup B \supseteq A \cup Y \cup B = X$.

This contradicts X being connected.

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Bonus question: What is $X/\emptyset$?

Hint: What is $\emptyset/\emptyset$?

Hint: What would Theorem 22.2 or Corollary 22.3 say? You might want to relax the surjectivity a bit.