24.1 (a) Q: Show $(0,1)$, $(0,1]$ & $[0,1]$ are not homeomorphic.

A: $\forall\, x \in (0,1)$ $(0,1)-\{x\}$ has two components. For $(0,1]$ there is one x s.t. $(0,1]-\{x\}$ is connected, namely $x=1$.
For $[0,1]$, there are two such x.
If $f: X \xrightarrow{\cong} Y$ is a homeomorphism, then $f|_{X-\{x\}}: X-\{x\} \to Y-\{f(x)\}$ is a homeomorphism for all $x \in X$.

(b) Q: If $f: X \to Y$ & $g: Y \to X$ are embeddings, show X & Y need not be homeomorphic.

A: $(0,1) \hookrightarrow [0,1]$ is an embedding.
$x \longmapsto x$
$(0,1] \longrightarrow (0,1)$ is an embedding
$x \longmapsto x/2$

By (a) they are not homeomorphic

(c) Q: Show $\mathbb{R}^n$ & $\mathbb{R}$ are not homeomorphic $n>0$.

A: $\mathbb{R}^n-\{x\}$ is connected $\forall\, x \in \mathbb{R}^n$.
$\mathbb{R}-\{x\}$ is disconn $\forall\, x \in \mathbb{R}$.
Same argument as in (a).

24.2 Q: $f: S^1 \to \mathbb{R}$, Show $f(x)=f(-x)$ some x.

A: Assume $f(0) < f(\pi)$. Consider
$f(x)-f(-x) = g(x) = f(x) - f(x+\pi)$. Then $g(0) < 0 < g(\pi)$.
Since $g(S^1)$ is connected, $g(x)=0$ for some $x \in S^1$ $\Rightarrow f(x)=f(-x)$

2

24.3 Q: Let $f: X \to X$. $X = [0,1]$. Show that $f(x) = x$ some $x \in X$.
What if $X = (0,1]$ or $[0,1)$

A: Consider $g(x) = f(x) - x$.
Then $g(0) \geq 0$ $g(1) \leq 0$. So
$0 \in [g(1), g(0)] \subseteq g(X)$.
If $x = (0,1]$ then $f(x) = x/2$ is a counterexample
If $x = [0,1)$ or $(0,1)$ then $f(x) = \cancel{2x}$ $\frac{1}{2} + \frac{1}{2}x$ is a counterexample.

24.8 (a) Q: Is a product of path conn. spaces path conn?

A: Yes. Let $x = (x_\alpha)$, $y = (y_\alpha) \in \prod X_\alpha$
Let $f_\alpha : I \to X_\alpha$ s.t. $f_\alpha(0) = x_\alpha$, $f_\alpha(1) = y_\alpha$
Then $f = (f_\alpha) : I \to \prod X_\alpha$ is a path from x to y.

(b) Q: If $A \subset X$ is path conn. is $\bar{A}$ path conn
A: No. The topologists sine curve is a counterexample, see example 7.

(c) Q: If $f: X \to Y$, X is path conn. is $f(x)$ path conn.
A: Yes. Let $f(x), f(y) \in f(X)$. Then $\exists \gamma: I \to$ s.t. $\gamma(0) = x$, $\gamma(1) = y$. Then $f \circ \gamma$ is a path from $f(x)$ to $f(y)$.

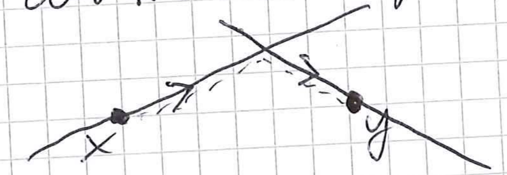
(d) Q: If $\{A_\alpha\}$ is a collection of path conn subspaces, of X, $\cap A_\alpha \neq \emptyset$. Is $\cup A_\alpha$ path conn
A: Yes. ~~[illegible]~~ Pick $x_0 \in \cap A_\alpha$. If $x, y \in \cup A_\alpha$, pick γ_1 a path from ~~x~~ to ~~x~~x_0, γ_2 path x_0 to y.

Then $\gamma : I \to X$, $\gamma(t) = \begin{cases} \gamma_1(t) & t \in [0, \frac{1}{2}] \\ \gamma_2(t) & t \in [\frac{1}{2}, 1] \end{cases}$

is a path from x to y. See Theorem 18.3 (pasting lemma).

24.9 Q: Assume $\mathbb{R}$ is uncountable. Let $A \subset \mathbb{R}^2$ be a countable subset. Show $\mathbb{R}^2 - A$ is path connected.

A: Given $x, y \in \mathbb{R}^2 - A$. Let L_x be the set of lines passing through x and no point in A. L_x is nonempty, and in fact uncountable, since A is countable. Indeed, if any line ~~in L_x~~ through x hit a point of A we could make a surjection $A \to$ ~~L_x. This con~~ {lines through x}. This contradicts the latter set being uncountable. Choose two non-parallel lines in L_x & L_y. Use these to construct a path:



24.10 Q: U open conn $\subseteq \mathbb{R}^2$. Show U is path con

A1: U is locally path connected, use theorem 25.

A2: Fix $x_0 \in U$. $C = \{x \in U \mid \exists$ path in U from x_0 to $x\}$

Then C is both open & closed in U.
Indeed, if $x \in C$, $\exists$ $B(x, \varepsilon) \subseteq U$, but C is path conn & $B(x, \varepsilon)$ is path conn & $C \cap B(x, \varepsilon)$ ~~is~~ $\neq \emptyset$, so 24.8(d) $\Rightarrow$ $B(x, \varepsilon) \cup C$ is path conn $\Rightarrow B(x, \varepsilon) \subseteq$ $\Rightarrow$ C is open. ~~Assume~~

4 Assume $y \notin C$. Then $\exists B(y, \varepsilon) \subseteq \mathcal{U}$ & $B(y, \varepsilon) \cap C = \emptyset$. If not there would be a path from x_0 to $x \in B(y, \varepsilon) \cap C$ & a pat from x to y, contradicting $y \notin C$. Hence C^c is open, so C is open & closed, so $C = \mathcal{U}$, since $\mathcal{U}$ is conn.

25.4 Q: X locally path conn. Show $\forall$ conn. open set in X is path conn.

A: Let $\mathcal{U}$ be open, conn $\subseteq X$. Let $x_0 \in \mathcal{U}$, $C = \{x \in \mathcal{U} \mid \exists$ path from x_0 to $x\}$.

Similar to 24.10

C is open, since if $x \in C$ $\exists V \subseteq \mathcal{U}$, $x \in V$

V is open & path conn $\Rightarrow V \subseteq C$.

C^c is open, since if $x \notin C$ $\exists V \subseteq \mathcal{U}$ $x \in V$

V is open & path conn $\Rightarrow V \cap C = \emptyset$.

Hence C is open & closed in $\mathcal{U}$, hence $C = \mathcal{U}$.

25.5 Q: Let $X = \mathbb{Q} \times 0 \cap [0,1] \times 0 \subseteq \mathbb{R}^2$. Let T = Union of all line segments from $0 \times 1 = p$ to points of X

(a) Show T is path connected, but is locally connected only at p.

A: T is the union of a collection of path connected spaces with a point p in common. Exercise 24.8(d) imply T is path connected.

T: 

Show any connected open neighbourhood of x must contain p, as was done in the lecture. Clearly not all open neighbourhoods of x contains p.

Argue as on the next page, but replace B(x, e) by your neighbourhood which does not contain p.

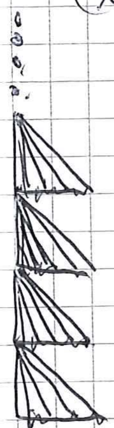
~~then there is no~~

5 Then x & $x' := (1-t)\,0\times 1 + t\left(y \pm \frac{1}{n} \times 0\right)$ (depending cavity $= 0\times 1$)

where $\frac{1}{n} < \varepsilon$
$\quad\;\; < y$

lie in different components of $B(x,\varepsilon)\cap T$.
Indeed, $B(x,\varepsilon)\cap T\cap C_1$, $B(x,\varepsilon)\cap T\cap C_2$ is a separation where C_1 & C_2 are the two components of $\mathbb{R}-l$, where l is a line from $y'\times 0$, to 0×1, where y' is an irrational number between y & $y\pm\frac{1}{n}$.
Any ball $B(x\times 0,\varepsilon)\cap T$ is connected, by the same argument as T is path connected.

(b) Q: Find a subset of $\mathbb{R}^2$ that is path conn but locally conn. at no point.



F

A: ~~Consider~~ Let T_n be $\{x\times(y+n) \mid x\times y \in T\}$

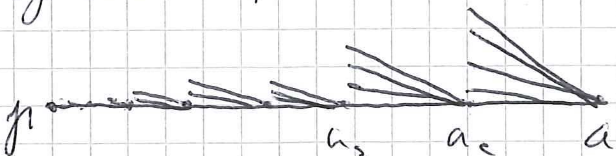
Define $F := \bigcup_{n\in\mathbb{N}} T_n$. Then F is path connected, but not locally connected at any point.

25.6 Q: X weakly locally connected at $x\in X$ iff $\forall\, U$ open nbhd of x $\exists$ V' conn $\subseteq U$, s.t. $\exists\, V$ open $\subseteq V'$ s.t. $x\in V$.
Show X is locally conn iff X is weakly locally conn at all its points.

A: We use Theorem 25.3: X is locally conn iff $\forall\, U\subseteq X$ each component of U is open. So give C a component of U, consider $x\in C$. Assumption $\Rightarrow \exists_{x\in} V'$ conn $\subseteq U \Rightarrow V'\subseteq C$, but $\exists_{x\in} V$ open $\subseteq V'$ $\subseteq C \Rightarrow \exists$ open nbhd around any point of $C \Rightarrow C$ is open

25.7 Q: Let X be the infinite broom.

X is made inductively ~~from~~ by taking all lines from $\frac{1}{n}\times 0$ to $\bigcup_{k\in\mathbb{Z}_+}\{\frac{1}{n+1}\times\frac{1}{n+k}\}\cup\{\frac{1}{n+1}\times 0\}$ and doing this for all $n\in\mathbb{Z}_+$.
Then you add $p=0\times 0$. X is considered as a subspace of $\mathbb{R}^2$. Note $a_n=\frac{1}{n}\times 0$.



Show X is weakly locally connected at p, but not locally connected at p.

A: Let U be a connected nbhd of p.
Then $\exists\, k$ s.t. $a_k\in U$. Assume $\exists\, n$ s.t. $a_{n+1}\in U$, $a_n\notin U$. Since U is open $\exists\, B(a_{n+1},\varepsilon)\subseteq U$. Then $B(a_{n+1},\varepsilon)$ intersects a line from a_n to ~~a_{n+}~~ $\frac{1}{n+1}\times\frac{1}{n+k}$ for some k. Then a_{n+1} & $\frac{1}{n+1}\times\frac{1}{n+k}$ cannot be in the same component of U, indeed, they are separated by $C_1\cap U$, $C_2\cap U$, where C_1, C_2 are the two components of $\mathbb{R}^2-l$, where l is the line through a_n & $\frac{1}{n+1}\times\frac{1}{2}\left(\frac{1}{n+k}+\frac{1}{n+k+1}\right)$.
This contradicts U being connected, so U must contain all a_n. Obviously not all nbhds of p contains all a_n, so X is not locally connected at p.
To prove U is weakly locally connected at p, observe that the closed ball $\bar{B}(p,\frac{1}{n})\cap X$ is connected (path connected), and contains the open ball $B(\cancel{a_{n+1}}, \frac{1}{n})\cap X$.