MAT 4500

26.5 Q: $A, B \subseteq X$, A, B compact $A \cap B = \emptyset$, X Hausdorff
$\Rightarrow \exists\ U, V$ open $\subseteq X$ s.t. $U \cap V = \emptyset$, $A \subseteq U$, $B \subseteq V$.

A: Fix $b \in B$. For any point $a \in A$ $\exists\ \overbrace{U_a, V_a}^{\text{open}}$ s.t. $b \in V_a$, $a \in U_a$ $U_a \cap V_a = \emptyset$, since X is Hausdorff.
Cover A by $\{U_a\}_{a \in A}$. Since A is compact
$\exists\ \underbrace{\{U_a\}_{a \in \{a_1, \dots, a_n\}}}_{\text{cover of } A}$ for some $\underbrace{a_1, \dots, a_n}_{=: A'_b} \in A$.
Then $b \in \bigcap_{a \in A'_b} V_a =: V^b$ & $U^b := \bigcup_{a \in A'_b} U_a \supseteq A$ are open.

Do this for any point $b \in B$. $\Rightarrow$ We get a cover $\{V^b\}_{b \in B}$ of B. Since B is compact $\exists\ B' = \{b_1 \dots b_m\} \subseteq B$ s.t. $\bigcup_{b \in B'} V^b =: V$ cover B. Then V is open and cover B & $U := \bigcap_{b \in B'} U^b$ is open & cover A & $U \cap V = \emptyset$.

26.6 Q: Show if $f: X \to Y$, cont, X comp., Y Hausdorff $\Rightarrow f$ is closed.

A: Let $A \subseteq X$ be a closed set $\overset{\text{Thm 26.2}}{\Rightarrow}$ A is compact $\overset{\text{Thm 26.5}}{\Rightarrow}$ $f(A)$ is compact $\overset{\text{Thm 26.3}}{\Rightarrow}$ $f(A)$ is closed.

26.7 Q: Y compact $\Rightarrow \pi_1 : X \times Y \to X$ is closed.

A: Let $A \subseteq X \times Y$, closed. Want to prove $\pi_1(A)^c$ is open. Consider $x \in \pi_1(A)^c \Rightarrow \{x\} \times Y \subseteq \pi_1^{-1}(\pi_1(A)^c)$
$\Rightarrow \{x\} \times Y \subseteq A^c \overset{\text{Tube lemma}}{\Rightarrow} \exists\ W$ open $\subseteq X$, $x \in W$ s.t. $W \times Y \subseteq A^c$
$\Rightarrow x \in \underbrace{\pi_1(W \times Y)}_{W} \subseteq \pi_1(A)^c \Rightarrow \pi_1(A)^c$ is open.

(of course, in general $\pi_1(A^c) \not\subseteq \pi_1(A)^c$, the above is only true because of the particular form of π_1 & $W \times Y$)

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26.8 Q: $f: X \to Y$, Y compact, T_2. ~~that~~ Show f is cont. iff $\Gamma_f := \{x \times f(x) \mid x \in X\}$ is closed in $X \times Y$.

A: ~~Assume A is closed $\subseteq Y$.~~

Assume Γ_f is closed. Let $A \subseteq Y$ be closed. Then $f^{-1}(A) = \pi_1(\Gamma_f \cap \pi_2^{-1}(A))$, which is closed by exercise 7. Hence f is continuous.

Assume f is continuous. Then $h: X \times Y \to Y \times Y$ given by $h(x \times y) = f(x) \times y$ is cont. $\Delta(Y) \subseteq Y \times Y$ is closed by exercise 17.13. since Y is T_2. ~~Furthermore~~ $h^{-1}(\Delta(Y)) = \Gamma_f$ is then closed.

27.4 Q: (X, d) metric space, connected. Show X is uncountable if $X \neq \{*\}$.

A: Fix $x \in X$. Then $d(x,-): X \to \mathbb{R}$ is cont. The image $d(x, X)$ is connected, and has at least two distinct points. Hence it contains an entire interval, which is uncountable. Hence we have a surjection from ~~$\mathbb{R}$~~ X to something uncountable, hence X is uncountable.

27.5 Q: X compact, T_2. $\{A\}_{n \in \mathbb{Z}_+}$ countable collection of closed sets s.t. $\operatorname{Int} A_n = \emptyset$. Show $\operatorname{Int} \bigcup A_n = \emptyset$.

A: Lemma: Given A closed, $\operatorname{Int} A = \emptyset$, U open $\neq \emptyset$. Then $\exists\, V \overset{\neq\emptyset}{\subset} U$ s.t. $\overline{V} \cap A = \emptyset$ & $\overline{V} \subset U$.

Proof: Consider $x \in U \setminus A \neq \emptyset$, since $\operatorname{Int} A = \emptyset$. $\Rightarrow$ $\exists\; U' \subseteq U$, $W \supseteq A$ s.t. $x \in U'$ & $U' \cap W = \emptyset$ U', W open. (cf. Lemma 26.4).

Then $\overline{U'} \cap A = \emptyset$. Consider the closed set $\overline{U'} \setminus U$. $\exists\, U'' \ni x$, $W' \supseteq \overline{U'} \setminus U$ s.t. $U'' \cap W' = \emptyset$, U'', W' open. (cf. lemma 26.4)
Let $V := U'' \cap U'$. Then $V \subset U'$, so $\overline{V} \cap A = \emptyset$.
Furthermore $\overline{V} \setminus U \subset \overline{U'} \setminus U$, but $y \in \overline{U'} \setminus U$ is not in $\overline{U''} \supseteq \overline{V}$ by construction, so $\overline{V} \setminus U = \emptyset$.
(This is essentially a (weak) reformulation of X being normal, cf. Theorem 32.3). $\boxed{}$

Proof of 27.5: Assume $\emptyset \neq U = \operatorname{Int} \bigcup A_n$.
Use lemma to inductively find V_n s.t. $V_n \subset V_{n-1}$, $\overline{V_n} \cap A_n = \emptyset$. Define $V_{-1} = U$.

Then $\overline{V_1} \supset \overline{V_2} \supset \cdots \Rightarrow \exists x \in \bigcap \overline{V_i} \neq \emptyset$, but $x \in \overline{V_i}\ \forall i \Rightarrow x \notin A_i\ \forall i \Rightarrow x \notin U$, A contradiction, since $x \in \overline{V_i} \subseteq U\ \forall i$.
(See also Theorem 48.2 for a slightly different approach).

27.6 Q: $A_0 := [0,1]$, $A_n := A_{n-1} - \bigcup_{k=0}^{\infty} \left(\frac{1+3k}{3^n}, \frac{2+3k}{3^n} \right)$

$C := \bigcap_{n \in \mathbb{Z}_+} A_n$, Cantor set.

(a) Show C is totally disconnected i.e. only connected subspaces are singletons.

A: Given X a subspace of C. Assume $x, y \in X$ $x < y$. Find k & n s.t. $x < \frac{1}{2}\left(\frac{1+3k}{3^n} + \frac{2+3k}{3^n} \right) < y$

Then $z \notin C$ & $U := X \cap (-\infty, z)$, $V \overset{!!}{:=} X \cap (z, \infty)$ form a separation of X.

(b) Q: Show C is compact.

A: Each A_n is closed, hence C is closed & bounded so C is compact (cf. Theorem 27.3).

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22.6 (c) Q: Show A_n is a finite disjoint union of closed intervals of length $1/3^n$. Show that the endpoints of the intervals lie in C.

A: $$A'_n := \bigsqcup_{k=0}^{3^{n-1}-3} \left[\frac{3k}{3^n}, \frac{1+3k}{3^n}\right] \sqcup \left[\frac{2+3k}{3^n}, \frac{3+3k}{3^n}\right]$$

$A'_0 := [0,1]$

Prove by induction $A_n = A'_n$.

The endpoints $\frac{k}{3^n}$ ~~are in~~ $0 \leq k \leq 3^n$ are in A_n for all n, hence in C.

(d) Q: Show C has no isolated point.

A: Assume $\{x\}$, $x \in C$ is open $\Rightarrow \exists\ \varepsilon$ s.t. $(x-\varepsilon, x+\varepsilon) \cap C = \{x\}$. But $\exists\ \frac{k}{3^n} \neq x$ s.t.

$k/3^n \in (x-\varepsilon, x+\varepsilon)$

contradicting $(x-\varepsilon, x+\varepsilon) \cap C = \{x\}$.

(e) Q: Show C is uncountable.

A: Theorem 22.7