MAT 4500 28.10

28.3 Q: X limit point compact

(a) $f: X \to Y$ is cont. is $f(X)$ limit point compact?

A = No, following example 1: $Y = \{1,2\}$ with trivial top. $\mathbb{Z}_+$, with discrete top. then $X = \mathbb{Z}_+ \times Y$ is limit point compact, $\pi_1 : \mathbb{Z}_+ \times Y \to \mathbb{Z}_+$ is cont, but $\pi_1(X) = \mathbb{Z}_+$ is not limit point compact. Indeed, $\mathbb{Z}_+$, has no limit points.

(b) Q: A closed $\subseteq X$ is A limit point compact?

A = Yes. Let Z be an infinite subset of A. Then Z has a limit point in X. This limit point lies in A, since A is closed (hence $\bar{Z} \subset \bar{A} = A$).

(c) Q: $X \subseteq Z$, Z is T_2, is X closed in Z?

A = This is true if Z has a countable basis at any point (i.e. first countable), but in general it is false. Indeed, following example 3, we have $S_\Omega \subset \bar{S}_\Omega$, $\bar{S}_\Omega$ is T_2 ~~that is well ordered~~ and S_Ω is limit point compact, but not closed in $\bar{S}_\Omega$.

29.1: Q: Show $\mathbb{Q}$ is not locally compact

A: Let C be a subspace of $\mathbb{Q}$ containing $(a,b)\cap\mathbb{Q}$ for some $a<b$. If C is compact, C is compact as a subspace of $\mathbb{R}$. Hence C is closed and bounded. But $C=\overline{C}\supset\overline{(a,b)\cap\mathbb{Q}}=[a,b]$, which is not contained in $\mathbb{Q}$, a contradiction. (An alternative approach is to construct an infinite cover of C with no finite subcover, using that (a,b) contains irrational numbers & $\mathbb{Q}$ is dense.)

29.2(a) Q: Show if $\prod X_\alpha$ is loc. comp. then X_α is loc. comp. & X_α is comp. for all but fin. many α.

A: Fix β. Given $x_\beta\in X_\beta$, ~~so~~ pick some $x\in\pi_\beta^{-1}(x_\beta)$. Since $\prod_\alpha X_\alpha$ is loc. comp. $\exists$ $C\subset\prod X_\alpha$ s.t. C is compact ~~&~~, $x\in C$ & $\exists\ U$ open, $x\in U$ & $U\subseteq C$. Then $x_\beta\in\pi_\beta(U)\subset\pi_\beta(C)$ is compact. $\exists\ U_\alpha\overset{\text{open}}{\subset}X_\alpha$ s.t. $x\in\prod_\alpha U_\alpha\subset U$

$\Rightarrow x_\beta\in\pi_\beta(\prod_\alpha U_\alpha)=U_\beta\subset\pi_\beta(U)\subset\pi_\beta(C)$

Hence each X_β is loc. comp. Furthermore $U_\alpha=X_\alpha$ for all but fin. many α, so $X_\alpha=\pi_\alpha(\prod_\alpha U_\alpha)\subset\pi_\alpha(C)$ is compact, i.e. X_α is compact, for all but fin. many α.

(b) Q: Prove the converse assuming Tychonoff's theorem

A: Write $\prod X_\alpha=\underset{X_\alpha\text{ is compact}}{\prod X_\alpha}\times\underset{X_\alpha\text{ is not compact}}{\prod X_\alpha}$ (finite product). Hence it suffices to show $X\times Y$ is loc. comp. if X & Y are loc. comp. This is easy: If $x\times y\in X\times Y$ $\exists\ U\overset{\text{open}}{\subseteq}C_1\overset{\text{comp}}{\subseteq}X$, $\exists\ V\overset{\text{open}}{\subseteq}C_2\overset{\text{comp}}{\subseteq}Y$ s.t. $x\in U\subseteq C_1\subseteq X$, $y\in V\subseteq C_2\subseteq Y$, but then $x\times y\in U\times V\subseteq C_1\times C_2\subseteq X\times Y$ & $C_1\times C_2$ is comp. & $U\times V$ is open.

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29.3 Q: X loc. comp. $f: X \to Y$ cont. Is $f(X)$ loc comp?

A: No. Any discrete space is loc. comp., but not all spaces are discrete. E.g. let $\mathbb{N} \to \mathbb{Q}$ be a bijection, where $\mathbb{N}$ has the discrete topology. The map is cont, but $\mathbb{Q}$ is not loc. comp.

~~DEN~~ Q: What if f is ~~cs~~ open & cont?

~~Let~~ $y \in f(X)$, $y = f(x)$ $\exists$ C comp., U open $\overset{\text{in } X}{\subseteq} C$

s.t. ~~X~~ $x \in U$. but then $y \in f(U)$ is open $\overset{\text{in } Y}{\subseteq} f(C)$ compact

29.6 Q: Show that the one point compactification of $\mathbb{R}$ is homeomorphic to S^1.

A: $\mathbb{R} \approx S^1 - \{(0,1)\}$ (e.g. stereographic projection $x \mapsto \left(\frac{2x}{x^2+1}, \frac{x^2-1}{x^2+1}\right)$). Homeomorphic spaces have homeomorphic one point compactifications.

S^1 is a 1-pt-compt. of $S^1 - \{(0,1)\}$, since

$S^1 - \{(0,1)\}$ is a subspace of S^1, with closure S^1

$S^1 - (S^1 - \{(0,1)\}) = \{(0,1)\}$ a single point

S^1 is compact T_2.

29.8 Q: Show 1-pt-compct. of $\mathbb{Z}_+$ is homeomorphic to ~~X~~ $Y = \{0\} \cup \{1/n \mid n \in \mathbb{Z}_+\} \subseteq \mathbb{R}$.

A: Embedd $\mathbb{Z}_+ \hookrightarrow \mathbb{R}$ via $n \mapsto \frac{1}{n}$.

Then ~~XX~~ Y is a ~~1~~-pt-compct. of $X = \{1/n \mid n \in \mathbb{Z}_+\}$ since

~~Wh~~ X is a subspace of Y, ~~and~~ $\overline{X} = Y$

~~X~~ $Y - X = \{0\}$

Y is compact T_2, since it is a closed & bounded subspace of $\mathbb{R}$.

$f: X \xrightarrow{\cong} X'$

Check this: Show that if $X \xrightarrow{\cong} X'$ is a homeomorphism then f extends to a homeomorphism of 1-pt compactifications $Y \to Y'$. map the extra points to each other.