30.3 Q: X has a countable basis, $A \subset X$ is uncountable. Show A has uncountably many limit points.

A: Assume A has countably many limit points $\{x_1, x_2, \ldots, x_n, \ldots\}$.
Then $A' = A^* = \{x_1, x_2 \ldots\} \subset X - \{x_1, x_2 \ldots\} = X'$ has no limit points (check this). We want to obtain a contradiction. We have reduced to A uncountable, no limit points. Hence, given $x \in X, x \notin A - \{x\}$, i.e. $\cap A - \{x\}$ $\cap U_n = \emptyset$ for some basis element $U_n \ni x$. This mean we can construct an injective function $f: A \to$ countable Basis of $X = \mathcal{B}$
Indeed, define $f(a) = U_n$ for some U_n s.t. $a \in U_n$, $U_n \cap (A - \{a\}) = \emptyset$. Then f is injective, since if $f(a) = f(a') = U_n$ then $a \in U_n \ni a'$, but $(A - \{a\}) \cap U_n = \emptyset$, so $a = a'$. f being injective contradicts that A is uncountable, since $\mathcal{B}$ is countable.

30.4 Q: X compact metric space. Show X has a countable basis.

A: $\forall n \; \exists \mathcal{A}_n$ finite cover of X by balls of diameter $< \frac{1}{n}$ (check this). Then $\mathcal{A} = \cup \mathcal{A}_n$ is a countable basis for X. Indeed, given $x \in X$ & an open nbhd U of x, $\exists$ n s.t. $B(x, 1/n) \subset U$.
$\exists B(y, \frac{1}{2n}) \in \mathcal{A}_{2n}$ s.t. $x \in B(y, \frac{1}{2n})$, since $\mathcal{A}_{2n}$ is a cover.
Then $x \in B(y, \frac{1}{2n}) \subseteq B(x, \frac{1}{n}) \subset U$, hence $\mathcal{A}$ is a basis.

30.10 Q: $X = \prod_{i \in \mathbb{N}} X_i$ $\forall i \exists A_i \underset{\text{countable}}{\overset{\text{dense}}{\subset}} X_i$

$\Rightarrow$ X has a countable dense subset A.

A: The obvious candidate for A, $\prod A_i$ is not countable. Fix $a^0 = (a_i^0) \in \prod A_i$.

Define $A = \left\{ a \in \prod A_i \mid \begin{array}{l} a_i = a_i^0 \text{ for all but} \\ \text{finitely many } i \end{array} \right\}$

Then A is countable (check this).

We need to check that A is dense:
Given ~~$x \in \prod X_i$~~ U an open set $\subset \prod X_i$
Then U contains $\prod U_i$ a basis element.
$U_i = X_i$ for all but fin. many i.
Then $A \cap \prod U_i \neq \emptyset$ & $A \cap U \neq \emptyset$.
Hence A is dense (cf. mandatory assignment)
Indeed, assume $U_i = X_i$ for $i > n$.
Pick $a_i \in U_i \cap A_i$ for $i < n$. Then
~~let~~ $a_i = a_i^0$ for $i \geq n$. Then $a = (a_i) \in A$
& $a \in \prod U_i \subset U$.

31.1 Q: X regular. Show $\forall x, y \in X$ $\exists$ nbhds
U, V of x, y s.t. $\overline{U} \cap \overline{V} = \emptyset$
A: Pick disjoint nbhd U' & V' of x & y. Then
$x \notin \overline{V'}$ since X is regular $\exists\, U \ni x$, $V' \supset \overline{V}$
s.t. $U \cap V' = \emptyset$, U, V' are open. Then $\overline{U} \cap \overline{V}$
$= \emptyset$, since $U \cap V' = \emptyset$.

31.2 Q: X normal. Show $\forall A, B$ closed, $A \cap B = \emptyset$, $\exists\, U, V$ open, $\overline{U} \cap \overline{V} = \emptyset$, $A \subseteq U$, $B \subset V$.

A: Pick disjoint open nbhds U', V of A & B. Then $A \cap \overline{V} = \emptyset$. Pick disjoint open nbhds U, V' of A & $\overline{V}$. Then $\overline{U} \cap \overline{V} = \emptyset$, & $U \supset A$, & $V \supset B$.

Usefull: Remember that a normal space is regular, a regular space is Hausdorff, and a Hausdorff space is T_1 (~~closed~~ points are closed).

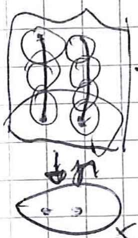
[When checking that a space is normal/regular, always remember to check that points are closed.

31.5 Q: $f, g: X \to Y$ cont. Y T_2. Show $\{x \mid f(x) = g(x)\}$ is closed in X.

A: The composition $h: X \xrightarrow{\Delta} X \times X \xrightarrow{f \times g} Y \times Y$ is continuous. The diagonal $\Delta_Y \subset Y \times Y$ is closed, since Y is T_2 (see exercise 17.13). Hence $h^{-1}(\Delta_Y) = \{x \mid f(x) = g(x)\}$ is closed.

31.7 Q: $p: X \to Y$ closed cont surj. s.t. $p^{-1}(\{y\})$ is compact $\forall y \in Y$ ("perfect map")

(a) Show Y is T_2 if X is T_2



A: Given $y_1, y_2 \in Y$, Then $p^{-1}(\{y_1\})$ & $p^{-1}(\{y_2\})$ are compact. Then $\exists\, U, V$ open, $U \cap V = \emptyset$ & $U \supseteq p^{-1}(\{y_1\})$ & $V \supseteq p^{-1}(\{y_2\})$, by a standard compactness argument. Then $p(U^c)^c \ni y_1$ is open & $p(V^c)^c \ni y_2$ is open & $p(U^c)^c \cap p(V^c)^c$ ~~is~~ $= \emptyset$. Indeed U^c is closed $\Rightarrow p(U^c)$ is closed, since p is closed $\Rightarrow p(U^c)^c$ is open. Similarly $p(V^c)^c$ is open.

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Furthermore $\pi(\mathcal{U}^c)^c \cap \pi(V^c)^c = (\pi(\mathcal{U}^c) \cup \pi(V^c))^c$

$= (\pi(\mathcal{U}^c \cup V^c))^c = \pi((\mathcal{U} \cap V)^c)^c = \pi(X)^c = Y^c = \emptyset.$

(b) Q: Show X regular $\Rightarrow$ Y regular

A: X regular $\Rightarrow X\ T_2 \overset{(a)}{\Rightarrow} Y\ T_2$, so Y is T_1.

Given A closed $\subseteq Y$, $y \in Y \Rightarrow \pi^{-1}(A)$ closed

$\pi^{-1}(\{y\})$ compact $\Rightarrow \exists\ \mathcal{U}$ open $\supseteq \pi^{-1}(A)$, V open $\supseteq \pi^{-1}(\{y\})$ s.t. $\mathcal{U} \cap V = \emptyset$, by compactness + X is regular.

Then $\pi(\mathcal{U}^c)^c \supseteq A$, $\pi(V^c)^c \supseteq \{y\}$ & $\pi(\mathcal{U}^c)^c \cap \pi(V^c)^c$ by the same argument as in a

(c) Q: X locally compact $\Rightarrow$ Y locally compact.

A: Given $y \in Y$. $\exists\ C$ compact $\supset \mathcal{U}$ open $\supseteq \pi^{-1}(\{y\})$

Indeed, $\forall x \in \pi^{-1}(\{y\})\ \exists\ C_x \overset{\text{comp}}{\ } \supset \mathcal{U}_x \overset{\text{open}}{\ } \ni x$, since $\pi^{-1}(\{y\})$ is compact, $\exists$ fin. many $\{C_{x_i} \supset \mathcal{U}_{x_i}\}_{i=1}^n$ s.t.

$\bigcup_{i=1}^n C_{x_i} \supset \bigcup_{i=1}^n \mathcal{U}_{x_i} \supset \pi^{-1}(\{y\})$. Then $\pi(C^c) \subset \pi(\mathcal{U}^c)$

$\Rightarrow \pi(\mathcal{U}^c)^c \subset \pi(C^c)^c \subset \pi(C)$, $\pi(C)$ is compact & $\pi(\mathcal{U}^c)^c$ is open & $y \in \pi(\mathcal{U}^c)^c$.

Note: $\pi(A^c)^c \subset \pi(A)$, since π is surjective.

Indeed $Y = \pi(X) = \pi(A \cup A^c) = \pi(A) \cup \pi(A^c)$

$= \pi(A) \setminus \pi(A^c) \cup \pi(A) \cap \pi(A^c) \cup \pi(A^c) \setminus \pi(A)$

$\Rightarrow \pi(A^c)^c = \pi(A) \setminus \pi(A^c) \subset \pi(A)$.

(d) Q: X second countable $\Rightarrow$ Y second countable

A: $\mathcal{B}$ countable basis for X. $\mathcal{B}' = \{\mathcal{U}_f \subseteq Y \mid$

$\mathcal{U}_f = \bigcup_{\pi^{-1}(W) \subset \bigcup_{V \in f} V} W$, f fin. subset of $\mathcal{B}$, W open $\}$

Then $\mathcal{B}'$ is a countable basis for Y. Indeed $\mathcal{B}'$ is countable, since $\mathcal{B}$ is (convince yourself of this

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Given $y \in Y$, & a nbhd U of y. Then $\pi^{-1}(U) \supset \pi^{-1}(\{y\})$. Cover $\pi^{-1}(\{y\})$ by a fin. subset $\mathcal{F}$ of $\mathcal{B}$, s.t. $\forall V \in \mathcal{F}$ $V \subset \pi^{-1}(U)$, i.e. $V_{\mathcal{F}} = \bigcup_{V \in \mathcal{F}} V \subset \pi^{-1}(U)$. Then $U_{\mathcal{F}} \subset U$.

Indeed, $\pi^{-1}(U_{\mathcal{F}}) \subset \pi^{-1}(U)$, by definition of $U_{\mathcal{F}}$

$$\Rightarrow \underbrace{\pi(\pi^{-1}(U_{\mathcal{F}}))}_{U_{\mathcal{F}}} \subset \pi(\pi^{-1}(U)) = U$$

claim: $y \in U_{\mathcal{F}}$. Indeed, $V_{\mathcal{F}}^c \supset \pi^{-1}(U)^c$

$$\Rightarrow \pi(V_{\mathcal{F}}^c) \supset \pi(\pi^{-1}(U)^c)$$

$$\Rightarrow y \in \underbrace{\pi(V_{\mathcal{F}}^c)^c}_{\text{open}} \subset \pi(\pi^{-1}(U^c))^c = U$$

Furthermore $\pi^{-1}(\pi(V_{\mathcal{F}}^c)^c) \subset V_{\mathcal{F}}$

$\Leftrightarrow \pi^{-1}(\pi(V_{\mathcal{F}}^c)) \supset V_{\mathcal{F}}^c$, which is true.

Hence $\exists U_{\mathcal{F}}$ s.t. $y \in U_{\mathcal{F}}$ & $U_{\mathcal{F}} \subset U$, i.e. $\mathcal{B}$ is a basis.

32.1 Q: $A \overset{\text{closed}}{\subset} X$, X normal $\Rightarrow A$ normal.

A: Given B, C, $\overset{\text{disjoint}}{\text{closed}} \subset A \Rightarrow B, C$ closed $\subset X$

$\Rightarrow \exists\, U, V$ open $\subset X$, s.t. $U \cap V = \emptyset$, $U \supset B$, $V \supset C$

$\Rightarrow U \cap A$, $V \cap A$ open in A s.t. $U \cap A \supset B$, $V \cap A \supset C$ & [illegible]

Note: X T_2 $\Rightarrow A$ T_2.

32.2 Q: $\prod X_\alpha$ is T_2, regular or normal $\Rightarrow X_\alpha$ is. $X_\alpha \neq \emptyset$ $\forall \alpha$

A: Imbedd X_α into $\prod X_\beta$: Fix $(x_\beta^0)_\beta \in \prod X_\beta$, then $f: X_\alpha \longrightarrow \prod X_\beta$, $x \longmapsto (x_\beta)_\beta$, where $x_\beta = x_\beta^0$ $\beta \neq \alpha$, $x_\alpha = x$. Then f is an imbedding. $f(X_\alpha) = \prod_{\beta \neq \alpha} \{x_\beta^0\} \times X_\alpha$ is a closed subspace of $\prod X_\alpha$. Use Theorem 31.2 $^{(a),(c)}$ on exercise 32.1 to conclude for T_2, regular or normal.

complement: $\bigcup_{\beta \neq \alpha} \left((X_\beta - \{x_\beta^0\}) \times \prod_{\beta' \neq \beta} X_{\beta'} \right)$