33.2(a) Q: X connected normal, $X \neq \{*\}$ $\Rightarrow$ X is uncountable.

A: $\exists\, x, y \in X$, $x \neq y$. Pick ~~a <~~ $b \in \mathbb{R}$.
Urysohns lemma $\Rightarrow \exists\, f: X \to [a,b]$, cont.
$f(x) = a$, $f(y) = b$. X connected $\Rightarrow f(X) = [a,b]$
$\Rightarrow$ X is uncountable.

(b) Q: X connected, regular, $X \neq \{*\} \Rightarrow$ X is uncountable

A: X countable $\Rightarrow$ X Lindelöf $\xRightarrow{\text{Exercise } 32.4}$ X normal
$\xRightarrow{(a) + \text{connected}}$ X uncountable

Proof of exercise 32.4 is a variant of Theorem 32.1.

33.3 Q: Prove Urysohns lemma for a metric space X.

A: Define $f(x) = d(x,A) / (d(x,A) + d(x,B))$.
Evidently $f(a) = 0 \ \forall\, a \in A$, $f(b) = 1 \ \forall\, b \in B$.
It suffices to show $d(x,A) + d(x,B) \neq 0 \quad \forall\, x$
& $d(x,A)$ is continuous.
$d(x,A) = 0$ iff $x \in A$. Indeed, $\Rightarrow B(x,\varepsilon) \cap A \neq \emptyset$
$\forall\, \varepsilon \Rightarrow x \in A$, since A is closed.
Given $y, x \in X \Rightarrow d(y,a) \leq d(y,x) + d(x,a)$
$$\Rightarrow \underbrace{\inf_{a \in A} d(y,a)}_{d(y,A)} \leq d(y,x) + \underbrace{\inf_{a \in A} d(x,a)}_{d(x,A)}$$
$\xRightarrow{\text{Symmetry}} |d(y,A) - d(x,A)| \leq d(y,x) \Rightarrow d(\cdot,A)$ is continuous.

33.7 Q: X locally compact $T_2 \Rightarrow$ X completely regular

A: X $T_2 \Rightarrow$ One point sets are closed (T_1)
Given $x_0 \in X$, $A \subseteq X$, A closed $x_0 \notin A$. $\exists\, U^{\text{open}} \subseteq X$
C compact $\subseteq X$, $x_0 \in U$, since X loc. comp. Since
C is compact T_2, we may assume $U \cap A = \emptyset$.
Urysohns lemma $\Rightarrow \exists\, f: C \to [0,1]$ s.t.

$f(x_0) = 1$, $f(C \setminus \mathcal{U}) = \{0\}$. (Since C is normal, cf. Theorem 32.3), Define $g: X \to [0,1]$, by $g(x) = f(x)$ $x \in C$. & $g(x) = 0$ $x \in \mathcal{U}^c$. Then g is continuous by the pasting lemma (Theorem 18.3). C is closed, see Theorem 26.3.

33.8 Q: X completely regular, A, B disjoint closed $\subseteq X$. A compact $\Rightarrow \exists\, f: X \to [0,1]$ s.t. $f(A) = \{0\}$, $f(B) = \{1\}$.

A: For every point $x \in A$ $\exists\, f_x: X \to [0,1]$ s.t. $f_x(x) = 1$, $f_x(B) = \{0\}$. $\{\mathcal{U}_x = f^{-1}((1/2, 1])\}$ is an open cover of A. Find a fin. subcover given by $\mathcal{U}_{x_1} \dots \mathcal{U}_{x_n}$. Define $f = \min(2\sum_{i=1}^{n} f_{x_i}, 1)$.

Then f is continuous, $f(B) = \{0\}$ and $f(a)$

$= \min(2\sum_{i=1}^{n} f_{x_i}(a), 1) = 1$, since $f_{x_i}(a) > \frac{1}{2}$ some i.

36.1 Q: M manifold $\Rightarrow M$ is regular.

A: Given $x \in M$, A closed $\subseteq M$ $x \notin A$. $\exists\, \mathcal{U} \overset{\text{open}}{\subseteq} M$ s.t. $\mathcal{U} \approx \mathbb{R}^n$, $x \in \mathcal{U}$. $\exists\, B(x, \varepsilon) \subset \mathcal{U}$

~~$B(x, \varepsilon/2) \subset B(x, \varepsilon)$. claim: $\overline{B(x, \varepsilon/2)} \subset B(x, \varepsilon)$.~~

This proof is not very nice

claim $\overline{B(x, \varepsilon)} \subset \mathcal{U}$. Assume $\exists\, y \in \overline{B(x, \varepsilon)}$ s.t. $y \notin \mathcal{U}$. $\exists\, V \ni y$ open $V \approx \mathbb{R}^m$. $\forall\, n$ $B(y, \frac{1}{n}) \cap B(x, \varepsilon) \neq \emptyset \Rightarrow \exists\, y_n \in B(y, \frac{1}{n}) \cap B(x, \varepsilon)$ $\{y_n\}$ has a convergent subsequence in $\mathcal{U}$. Let say it converges to y'. At the same time it converges to y. Since M is $T_2 \Rightarrow y = y'$, but $y \notin \mathcal{U} \ni y'$, a contradiction. Hence $\overline{B(x, \varepsilon)} \subset \mathcal{U}$, Pick ε s.t. $B(x, \varepsilon) \cap A = \emptyset \Rightarrow \overline{B(x, \varepsilon/2)}$ $\cap A = \emptyset \Rightarrow x \in B(x, \varepsilon/2)$, $A \subseteq \overline{B(x, \varepsilon/2)}^c$.

36.2 Q: X compact T_2. $\forall x \in X$ $\exists$ nbhd U of x, $k \geq 0$ s.t. U imbedds ~~into~~ $\mathbb{R}^k$. Show X imbedds in $\mathbb{R}^N$ for some $N \geq 0$.

A: This is essentially Theorem 36.2. Replace $g_i: U_i \to \mathbb{R}^m$ by $g_i: U_i \to \mathbb{R}^{k_i}$, ~~& modify~~ throughout the proof.

36.3 Q: X compact T_2 $\forall x \in X$ $\exists$ nbhd U s.t. $U \simeq \mathbb{R}^m$. Show X is an m-manifold.

A: It suffices to show X has a countable basis. Cover X by $U_1 \ldots U_n$, s.t. $U_i \simeq \mathbb{R}^m$ $\forall i$. Then each U_i has a countable basis, hence X has a countable basis.

Note: In the above exercises I have defined a manifold to be ~~a~~ space, where each point x has a nbhd U s.t. $U \simeq \mathbb{R}^m$ + extra conditions. This is equivalent to the definition in Munkres, since $B(x, \varepsilon) \simeq \mathbb{R}^m$.

36.5. Q: $X = \mathbb{R} - \{0\} \cup \{p, q\}$. $\mathcal{T}_X^{\text{basis}} = \{U \text{ open} \subset \mathbb{R} - 0\}$ $\cup \{(-a, 0) \cup \{p\} \cup (0, a)\} \cup \{(-a, 0) \cup \{q\} \cup (0, a)\}$ $\forall a > 0$. (Equivalently $X = \mathbb{R}_1 \sqcup \mathbb{R}_2 / x_1 \sim x_2$ iff $x_1 = x_2$ & $x_1 \neq 0 \neq x_2$)

(a) check this is a basis

A: 1. Any $x \in X$ $\exists$ basis element containing x. This is clear.

2. $x \in B_1 \cap B_2 \Rightarrow \exists$ $B_3 \subset B_1 \cap B_2$ & $x \in B_3$.

~~Example: If $x \in (-a, 0) \cup \{p\} \cup (0$~~

The intersection of two basis elements are always a basis element (check this) (This is special for this basis)

(b) Q: Show $X - \{p\} \simeq \mathbb{R}$ & $X - \{q\} \simeq \mathbb{R}$

A: Define $f: X - \{p\} \to \mathbb{R}$, by $f|_{\mathbb{R}-0} = \text{inc}_{\mathbb{R}-0}: \mathbb{R}-0 \to \mathbb{R}$.
~~$f(p) = 0$~~

4 ~~Then~~ & $f(q)=0$. Then f maps basis elements to open sets of $\mathbb{R}$, & the inverse image of an open interval in $\mathbb{R}$ is a basic element. f is clearly a bijection. Hence f is a homeomorphism.

(c) Let $y, x \in X$. If ~~$x \in \mathbb{R}-0$~~ $x \& y \in \mathbb{R}-0$, then ~~$(x-\varepsilon, x)$, $(x-1, y)$, y~~ $(x-\varepsilon, x+\varepsilon)$ & $(y-\varepsilon, y+\varepsilon)$ are disjoint for some ε. If $y = p \vee q$, then $(x-\varepsilon, x+\varepsilon)$ & $(-\varepsilon, 0) \cup \{y\} \cup (0, \varepsilon)$ are disjoint for some ε. Hence $\overline{\{x\}} = \{x\}$ $\forall x \in \mathbb{R}-0$.

Any nbhd of $\{p\}$ ~~& $\{x\}$~~ does not contain q. Hence $\overline{\{p\}} = \{p\}$. Hence X is T_1.

X is not T_2, since any nbhd of p & q intersect.

(d) $X - \{p\}$ is open & homeomorphic to $\mathbb{R}$.

$X - \{q\}$ is open & homeomorphic to $\mathbb{R}$.

$\mathbb{R}$ has a countable basis.

Since $X = (X - \{p\}) \cup (X - \{q\})$, X has a countable basis, and satisfies all the properties of a 1-manifold, except being T_2