1 MAT 4500 18.11.16

51.1 Q: Show if $h, h' : X \to Y$ are homotopic, & $k, k' : Y \to Z$ are homotopic, the $k \circ h$ & $k' \circ h'$ are homotopic.

A: Let $H_1 : X \times I \to Y$ be a homotopy between h & h'.
Let $H_2 : Y \times I \to Z$ be — " — k & k'.
Then $H_3 : X \times I \xrightarrow{(H_1, t)} Y \times I \xrightarrow{H_2} Z$ is a homotopy between $k \circ h$ & $k' \circ h'$. Indeed, H_3 is continuous.
$H_3(x, t) = H_2(H_1(x, t), t)$.
Then $H_3(x, 0) = H_2(H_1(x, 0), 0) = H_2(h(x), 0) = kh(x)$
, while $H_3(x, 1) = H_2(H_1(x, 1), 1) = k'(H_1(x, 1)) = k'h'(x)$.
Hence H_3 is a homotopy between kh & $k'h'$.

51.2 Q: X, Y spaces, $[X, Y]$ = set of homotopy classes ~~between~~ of maps between X & Y.

(a) Show $[X, I] = \{*\}$, when $I = [0, 1]$

A: We show any map $f : X \to I$ is homotopic to the constant map $c_0 : X \to I$, $c_0(x) = 0$.
Define ~~$H(x,t)$~~ $H : X \times I \to I$, by $H(x, t) = t f(x)$
Then $H(x, 0) = 0 \cdot f(x) = 0 = c_0(x)$, while
$H(x, 1) = 1 \cdot f(x) = f(x)$. Hence $f \sim c_0$.

(b) Q: Y path connected $\Rightarrow [I, Y] = \{*\}$.
Given $f : I \to X$, define a homotopy from f to a constant map. $H : I \times I \to X$, $H(s, t) = f(s(1-t))$
Then $H(s, 0) = f(s)$, while $H(s, 1) = f(0)$.
~~Given~~ Fix $y_0 \in Y$. Given $y \in Y$ Define a homotopy from the constant map $c_{y_0} : I \to Y$ to the constant map $c_y : I \to Y$, by choosing a path $\gamma : I \to Y$ from y_0 to y. Define $H : I \times I \to Y$ by $H(s, t) = \gamma(t)$. Then $H(s, 0) = c_{y_0}$, while $H(s, 1) = c_y$.

51.3 Q: X contractible if $id_X : X \to X$ is nullhomotopic

(a) Show I & $\mathbb{R}$ are contractible

A: $H : X \times I \to X$, defined by $H(s,t) = st$ defines a homotopy between c_0 & id_X for $X = I \vee \mathbb{R}$.

(b) Q: Show a contractible space is path connected

A: Let $H : X \times I \to X$ be a homotopy between c_{x_0} & id_X. Then $\gamma_x : I \to X$, $\gamma_x(t) = H(x,t)$ is a path from $x_0 = H(x,0)$ to x.

(c) Q: Show Y contractible $\Rightarrow [X,Y] = \{*\}$.

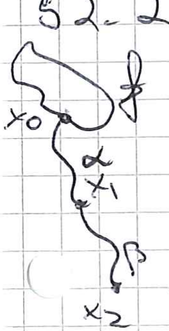
A: Let $H : Y \times I \to Y$ be a homotopy between c_{y_0} & id_Y. Given $f : X \to Y$, there is a homotopy from $c'_{y_0} : X \to Y$ to f defined by $H' : X \times I \to Y$, $H'(x,t) = H(f(x),t)$, i.e. $H' = X \times I \xrightarrow{(f,t)} Y \times I \xrightarrow{H} Y$. Hence $[X,Y] = \{*\}$.

(d) Q: Show Y path conn. X contractible $\Rightarrow$ $[X,Y] = \{*\}$

A: Let $H : X \times I \to X$ be a homotopy between c_{x_0} & id_X. Given $f : X \to Y$, there is a homotopy between $c_{f(x_0)} : X \to Y$ & f, defined by $H' : X \times I \to Y$, $H'(x,t) = f(H(x,t))$. Hence any map is homotopic to a constant map. Argue as in (b) to conclude $[X,Y] = \{*\}$

52.2 Q: $\alpha : I \to X$ path from x_0 to x_1

$\beta : I \to X$ path from x_1 to x_2



Show $\gamma = \alpha * \beta \Rightarrow \hat{\gamma} = \hat{\beta} \circ \hat{\alpha}$.

A: We have to show $\hat{\gamma}([f]) = \hat{\beta}(\hat{\alpha}([f]))$ $\forall [f] \in \pi_1(X, x_0)$. We have $\hat{\gamma}([f]) = [\bar{\gamma}] * [f] * [\gamma]$
$= [\bar{\beta} * \bar{\alpha}] * [f] * [\alpha * \beta] = [\bar{\beta}] * [\bar{\alpha}] * [f] * [\alpha] * [\beta]$
$= [\bar{\beta}] * \hat{\alpha}([f]) * [\beta] = \hat{\beta}(\hat{\alpha}([f]))$.

52.3 Q: $x_0, x_1 \in X$, path conn. Show $\pi_1(X, x_0)$ is abelian $\Leftrightarrow$ $\forall\, \alpha, \beta$ paths from x_0 to x_1 then $\hat{\alpha} = \hat{\beta}$.

"$\Leftarrow$"

A: Let α be a path from x_0 to x_1. Given $\gamma_1, \gamma_2 \in \pi_1(X, x_0)$.

Then $\hat{\alpha} = \widehat{\alpha * \gamma_1}$ $\Rightarrow$ $\hat{\alpha}(\gamma_2) = \widehat{\alpha * \gamma_1}(\gamma_2)$

$\Rightarrow \bar{\alpha} * \gamma_2 * \alpha = \bar{\alpha} * \bar{\gamma}_1 * \gamma_2 * \gamma_1 * \alpha$

$\Rightarrow \gamma_2 = \bar{\gamma}_1 * \gamma_2 * \gamma_1$, i.e. $\pi_1(X, x_0)$ is abelian.

"$\Rightarrow$" $\bar{\alpha} * \gamma_1 * \alpha = \bar{\beta} * \gamma_1 * \beta \Leftrightarrow \beta * \bar{\alpha} * \gamma_1 * \alpha * \bar{\beta} = \gamma_1$

& $\beta * \bar{\alpha} \in \pi_1(X, x_0) \ni \alpha * \bar{\beta}$, If $\pi_1(X, x_0)$ is abelian

$\beta * \bar{\alpha} * \gamma_1 * \alpha * \bar{\beta} = \gamma_1 * \underbrace{\beta * \bar{\alpha} * \alpha * \bar{\beta}}_{e_{x_0}} = \gamma_1$

Note: I am a bit sloppy with the $*$, one might want to distinguish between the product in $\pi_1(X, x_0)$, and concatenation of paths.

52.4 Q: $A \subset X$, $r: X \to A$, cont. s.t. $r(a)$ $\forall a \in A$. $a_0 \in A$. Show $r_*: \pi_1(X, a_0) \to \pi_1(A, a_0)$.

A: Let $j: A \to X$ ~~be~~ the inclusion. Then $r \circ j = id_A$. Hence $r_* \circ j_* = (id_A)_*$, so r_* has a right inverse, hence is surjective. (i.e. given $\gamma_1 \in \pi_1(A, a_0)$, consider $j_*(\gamma_1) \in \pi_1(X, x_0)$, then $r_*(j_*(\gamma_1)) = (r \circ j)_*(\gamma_1) = id_*(\gamma_1) = \gamma_1$, so r_* is surj.)

52.6 Q: X path conn. $h: X \to Y$, cont. $h(x_0) = y_0$, $h(x_1) = y_1$, α path x_0 to x_1. $\beta = h \circ \alpha$, path y_0 to y_1. Show $\hat{\beta} \circ (h_{x_0})_* = (h_{x_1})_* \circ \hat{\alpha}$

A: $\hat{\beta} \circ (h_{x_0})_*([\gamma]) = \hat{\beta}([h \circ \gamma]) = \bar{\beta} * [h \circ \gamma] * \beta$

$= \overline{(h \circ \alpha)} * [h \circ \gamma] * (h \circ \alpha) = h_*([\bar{\alpha} * \gamma * \alpha])$

$= h_*(\hat{\alpha}(\gamma))$.

22.1 Q: H Top. space, H is T_1 space, H is group. Show H top. grp $\Longleftrightarrow$ $f: H\times H \to H$, $x, y \mapsto xy^{-1}$ is cont.

A: Assume f is cont. Then $g: H \to H$ defined by $g(x) = f(e,x) = x^{-1}$ is cont. Then ~~$H\times H$~~

$H\times H \xrightarrow{id \times g} H \times H \xrightarrow{f} H$ is cont. & equals the multiplication on H.

Assume H is a top. grp. $\Rightarrow$ $f: H\times H \xrightarrow{id\times g}$

$H\times H \xrightarrow{\cdot} H$ is cont.

22.2 Q Show the following are top. grps

(a) $(\mathbb{Z}, +)$ A: $\mathbb{Z}$ has the discrete topology, so any map is continuous, in particular $x, y \mapsto x-y$.

(b) $(\mathbb{R}, +)$ A: We check ε-δ-continuity. Given ε, x, y & $x\times y \in \mathbb{R}^2$, then $\overset{d(f(x),f(y))}{|(x-y)-(x'-y')|} \leq |x-x'| + |y-y'|$ $= d_{\mathbb{R}^2}(x\times y, x'\times y')$, where $\mathbb{R}^2$ has the manhattan metric

Hence $d_{\mathbb{R}}(f(x\times y), f(x'\times y')) < \varepsilon$, when $d_{\mathbb{R}^2}(x\times y, x'\times y') < \delta = \varepsilon$.

(c) $(\mathbb{R}_+, \cdot)$ A: Given $\varepsilon > 0$, $x\times y \in \mathbb{R}^2$, ~~then~~ let $x'\times y' \in \mathbb{R}^2$, $d_{\mathbb{R}^2}(x\times y, x'\times y') < \delta$.

$$d(f(x\times y), f(x'\times y')) = \left|\frac{x}{y} - \frac{x'}{y'}\right| = \left|\frac{x}{y} - \frac{x}{y'} + \frac{x}{y'} - \frac{x'}{y'}\right|$$

$$\leq |x| \cdot \left|\frac{y'-y}{yy'}\right| + \frac{1}{|y'|} \cdot |x-x'|, \quad \left(\text{assume } \delta < \frac{y}{2}\right.$$

$$\left.\text{i.e. } |y-y'| < \frac{y}{2} \Rightarrow \frac{y}{2} < y' \right)$$

$$\Rightarrow \leq 2|x| \cdot \frac{|y'-y|}{|y|^2} + \frac{2}{|y|} \cdot |x-x'|$$

Pick δ s.t. $\frac{2|x|}{|y|^2} \cdot \delta < \frac{\varepsilon}{2}$ & $\frac{2}{|y|} \cdot \delta < \frac{\varepsilon}{2}$, $\delta < \frac{y}{2}$

$\Rightarrow d_{\mathbb{R}}(f(x\times y), f(x'\times y')) < \varepsilon$, when $d_{\mathbb{R}^2}(x\times y, x'\times y') < \delta$.

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(d) $(S^1, \cdot)$ A: Then $f((a,b),(c,d)) = (ac-bd, ad+bc)$, which is the restriction of compositions of functions defined in (b) & (c) (The components of f are polynomials.)

(e) $(GL(n), \cdot)$ A: Yet again the components of f are polynomials in the variables. Indeed, Cramers rule gives $(A^{-1})_{ij} = \frac{(-1)^{i+j} M_{ji}}{\det A}$, where $M_{ij} = i,j$ - minor of A,

i.e. $\det{}^{i}\left(\begin{array}{c|c} & \\ \hline & \end{array}\right) = M_{ij}$, determinant of A with row i & column j erased. (It is an entertaining exercise to check this. It only relies on properties of the determinant.)

Since $(AB)_{ij} = \sum_k A_{ik} B_{kj}$, the multiplication & "take inverse" map are continuous, hence $GL(n)$ is a topological group.

22.4 Q: Let $\alpha \in G$, show $f_\alpha(x) = \alpha \cdot x$, & $g_\alpha(x) = x \cdot \alpha$ are homeomorphisms of G.

A: $f_\alpha(x) = \mu(\alpha, x)$, i.e. $f_\alpha = G \longrightarrow G \times G \xrightarrow{\mu} G$, $y \longmapsto \alpha \times y$, $x \times y \longmapsto x \cdot y$, hence f_α is continuous. $f_\alpha^{-1} = f_{\alpha^{-1}}$, hence f_α is a homeomorphism

22.5 Q: $H < G$. $x \in G$, define $xH = \{x \cdot h \mid h \in H\}$ left coset. $G/H = \{xH \mid x \in G\}$, with quotient topology.

(a) Show, $\alpha \in G$, f_α induces a homeomorphism of G/H, mapping xH to $(\alpha x)H$.

A: Consider the map $G \xrightarrow{f_\alpha} G \xrightarrow{\pi} G/H$. Theorem 22.2. Gives us an induced map $\tilde{f}_\alpha : G/H \to G/H$ s.t. $\tilde{f}_\alpha \pi = \pi f_\alpha$. In fact corollary 22.3 implies $\tilde{f}_\alpha$ is a homeomorphism.

Q: (b) Assume H is closed. Show G/H is T_1.

A: $xH \in G/H$ is closed $\Leftrightarrow$ $\pi^{-1}(\{xH\}) = f_x(H)$ is closed. But the latter set is closed, since f_x is a homeomorphism & H is closed.

(c) Q: Show $\pi: G \to G/H$ is open.

A: Given U open $\subseteq G$. Then $\pi(U)$ is open $\iff \pi^{-1}(\pi(U))$ is open, but $\pi^{-1}(\pi(U)) = \bigcup_{h \in H} f_h(U)$, and $f_h(U)$ is open, since f_h is a homeomorphism, hence $\pi^{-1}(\pi(U))$ is open, since unions of opens are open.

(d) Q: H closed, $H \trianglelefteq G \Rightarrow G/H$ is a topological grp. ⊠

A: By (b) G/H is T_1. It remains to show multiplication & taking inverses are continuous. This follows from Theorem 22.3, since multiplication on G/H is induced by

$$\begin{tikzcd} G \times G \arrow[r] \arrow[d, "\pi \times \pi"'] & G \arrow[r, "\pi"] & G/H \\ G/H \times G/H \arrow[urr, dashed] & & \end{tikzcd}$$

while taking inverses are induced by

$$\begin{tikzcd} G \arrow[r, "-1"] \arrow[d, "\pi"'] & G \arrow[r, "\pi"] & G/H \\ G/H \arrow[urr, dashed] & & \end{tikzcd}$$

Note: The product of quotient maps are not necessarily a quotient map (see exercise 22.6). However, $\pi \times \pi$ is a quotient map, since π, hence also $\pi \times \pi$, is an open map.

Note: Any topological groups, as Munkres defines them, are T_2, since they are T_1, $f(x,y) = xy^{-1}$ is continuous, and hence $f^{-1}(\{e\}) = \Delta_G \subseteq G \times G$ is closed.