1 28.11.16 MAT4500

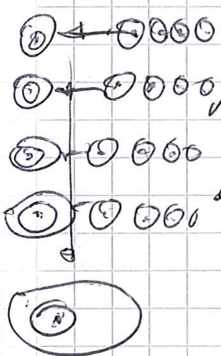
53.3 Q: $p: E \to B$ covering map, B conn. $p^{-1}(b_0)$ has k elements $\Rightarrow p^{-1}(b)$ has k elements $\forall b \in B$.

A: Let $U = \{b \in B \mid p^{-1}(\{b\})$ has k elements$\}$

Then U is open & closed, ~~and~~ nonempty ($b_0 \in U$). Hence $U = B$. Indeed, given $b \in U$, $\exists\, U_b$ nbhd of b s.t. $p^{-1}(U_b) = \bigsqcup V_{b_i}$ & $V_b \xrightarrow[\approx]{p} U_b$. Then $\forall b' \in U_b$ $p^{-1}(b')$ has k elements.

Conversely, given $b \notin U$, $\exists\, U_b$ s.t. $p^{-1}(U_b) = \bigsqcup V_b$, $V_b \xrightarrow[\approx]{p} U_b$. Hence $\forall b' \in U_b$ $p^{-1}(b')$ does not have k elements.

53.4 Q: Let $q: X \to Y$, $r: Y \to Z$ covering maps, $p = r \circ q$, $r^{-1}(z)$ is finite $\forall z \in Z \Rightarrow p$ is a covering map.



A: p is surjective, since r & q are. Given $z \in Z$, $r^{-1}(z) = \{y_1, \dots, y_n\}$, $\exists_z\, U^{\text{open}}$ s.t. $r^{-1}(U) = \bigsqcup_{i=1}^{n} V_i$, $y_i \in V_i$, ~~& $\forall \alpha$~~ $\exists_{y_i}\, V_i'^{\text{open}} \subset V_i$ s.t. $q^{-1}(V_i') = \bigsqcup_\alpha W_\alpha^i$ & $V_i \xrightarrow[\approx]{r} U$ & $W_\alpha^i \xrightarrow[\approx]{q} V_i'$. Define $V = \bigcap_{i=1}^{n} r(V_i')$.

Then V is open, $z \in V$, $r^{-1}(V) = \bigsqcup V_i''$, $V_i'' \xrightarrow[\approx]{r} V$ ($V_i'' = r^{-1}(V) \cap V_i$). Furthermore $q^{-1}(V_i'') = \bigsqcup W_\alpha^i \cap q|_{W_\alpha^i}^{-1}(V_i'') = \bigsqcup W_\alpha^{i'}$, where $W_\alpha^{i'} = W_\alpha^i \cap q|_{W_\alpha^i}^{-1}(V_i'')$

& $W_\alpha^{i'} \xrightarrow[\approx]{r \circ q} V$ & $(r \circ q)^{-1}(V) = \bigsqcup_{\alpha, i} W_\alpha^{i'}$

(Main idea: We have to shrink the nbhd around z, which we can do, since $r^{-1}(z)$ is finite)

5.4.4 Q: $p: \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}^2 - 0$, $p(x,y) = (y\cos x, y\sin x)$
is a covering map. Lift the following paths:

$f(t) = (2-t, 0)$

A: $\tilde{f}(t) = (0, 2-t)$

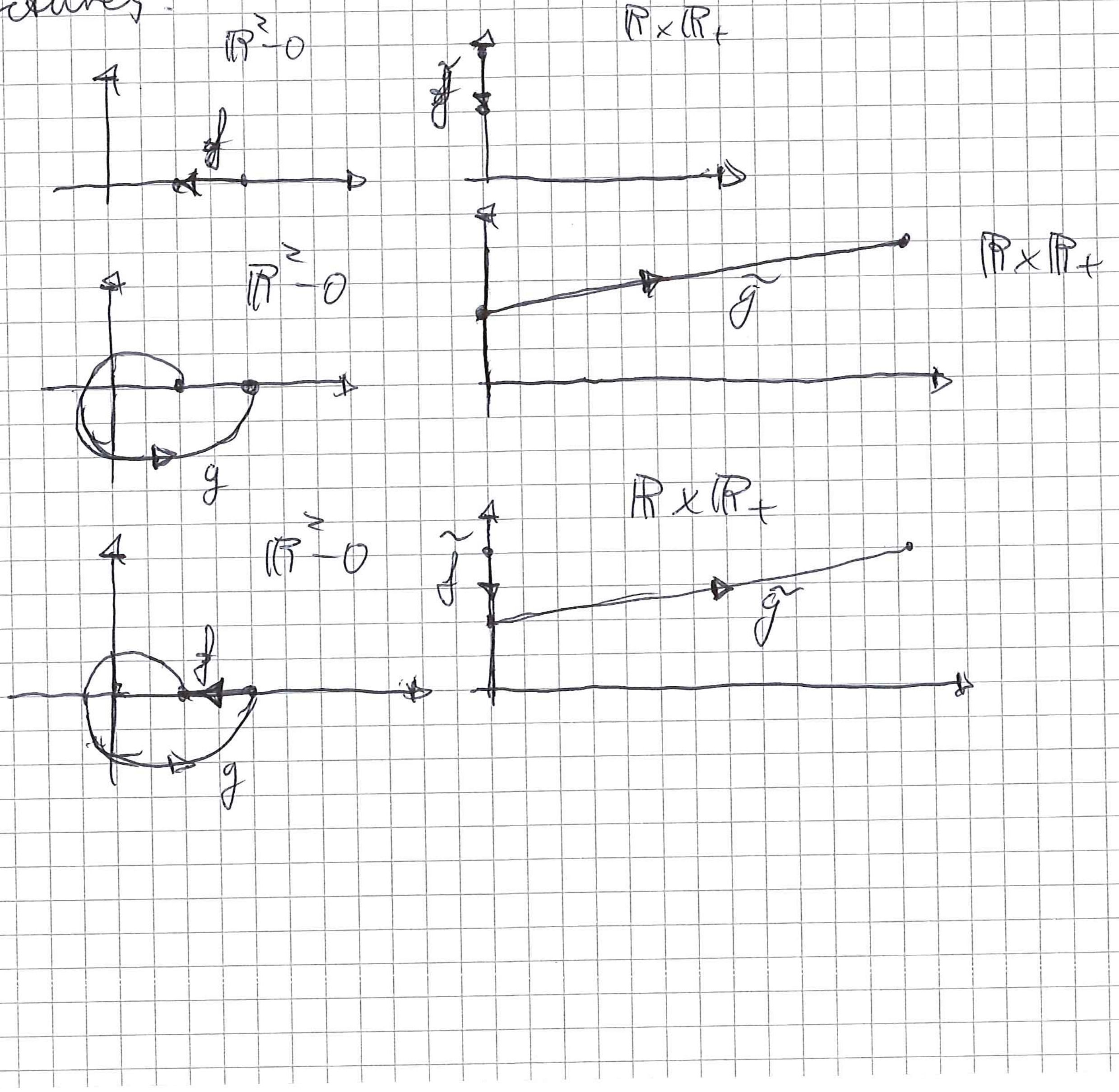
Q: $g(t) = ((1+t)\cos 2\pi t, (1+t)\sin 2\pi t)$

A: $\tilde{g}(t) = (2\pi t, 1+t)$

Q: $f * g$

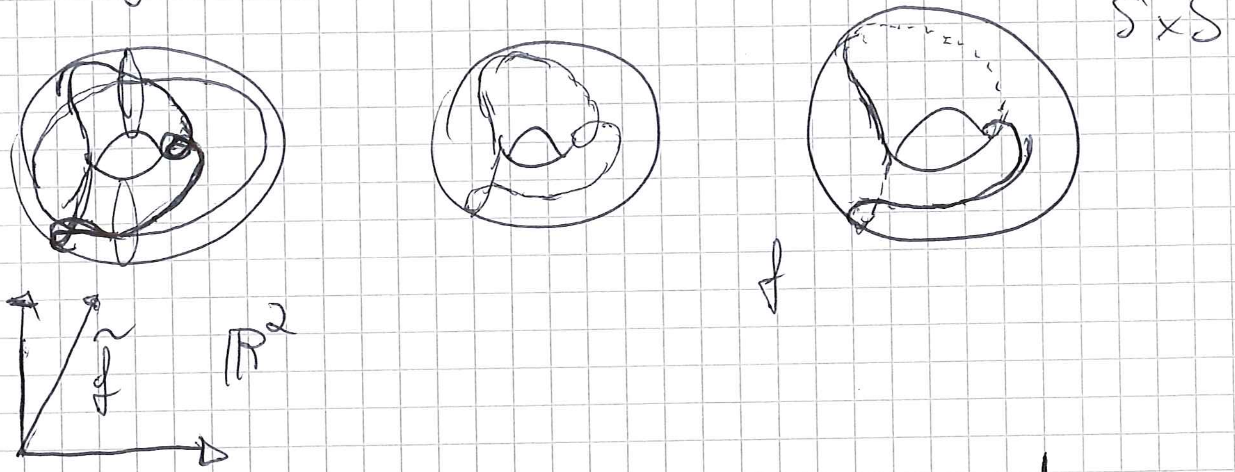
A: $\tilde{f} * \tilde{g}$ (Note: Another lifting for f is $(2\pi, 2-t)$, in which case this answer would not work).

Pictures:



54.5 Q: $p\times p: \mathbb{R}\times\mathbb{R}\to S^1\times S^1$, $(s,t)\mapsto(e^{it}, e^{is})$

A: $\tilde{f}(t)=(2\pi t, 4\pi t)$



54.6 Q: $g,h: S^1\to S^1$, $g(z)=z^n$, $h(z)=\frac{1}{z^n}$

Compute $g_*, h_*: \pi_1(S^1,b_0)\to$.

A: By Theorem 54.5 a generator for $\pi_1(S^1,b_0)$ is $\gamma: t\mapsto$ ~~cos~~ $e^{i2\pi t}$. I.e. $\phi([\gamma])=1$. Furthermore ~~$n[\gamma]$~~ $n[\gamma]=[t\mapsto e^{i2\pi nt}]$. But $g_*([\gamma]) = t\mapsto g(e^{i2\pi t}) = t\mapsto e^{i2\pi nt} = n[\gamma]$, and similarly for h_*. Hence g_* is multiplication by n, while h_* is multiplication by $-n$.

55.1 Q: A retract of $B^2 \Rightarrow \forall f: A\to A$ has a fixed point

A: Brouwer fixed point theorem (Theorem 55.6) implies any self map of B^2 has a fix point. In particular $ifr: B^2\to B^2$ has a fix point (here $i: A\to B^2$ is the inclusion). i.e. $ifr(x)=x$ some $x\in B^2$, but then $x\in A \Rightarrow f(x)=x$.

55.2 Q: $h: S^1 \to S^1$ is nullhomotopic $\Rightarrow$ $\exists\, x$ s.t. $h(x) = x$, $\exists\, y$ s.t. $h(y) = -y$.

A: Assume h has no fix/antipodal point.
Define $H: S^1 \times I \to S^1$, $H(x,t) = \dfrac{h(x)t \pm (1-t)x}{\| h(x)t \pm (1-t)x \|}$

Check that H is well defined.
Then H defines a homotopy from $\pm id_{S^1}$ to h. But id_{S^1} is not nullhomotopic. A contradiction

55.3 Q: A nonsingular 3×3 matrix with non-negative entries. Show A has a positive real eigenvalue.

A: This is essentially corollary 55.7, just replace positive by negative, and use that A is nonsingular to conclude $T(x)$ has a positive entry for any $x \in B$.