Exercises MAT4500 2/12-16.

58 : 2
59 : 1, 2
60 : 1, 2, 3, 4, 5

Recall:

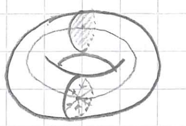
- Retraction of a map $j: A \to X$ is a map $r: X \to A$ such that $r \circ j = id_A$. (left inverse of j)
- Deformation retraction (for $A \overset{j}{\subseteq} X$ subspace): Retraction $r: X \to A$ + homotopy $j \circ r \simeq id_X$ which is stationary on A ("homotopy relative to A"), i.e: Homotopy $H: X \times I \to X$ s.t.

$$H(x,0) = x, \quad \overset{= r(x)}{H(x,1)} \in A \quad \text{and}$$
$$H(a,t) = a \ \forall a \in A.$$

[So $r \circ j = id_A$, $j \circ r \simeq id_X$].

In exercise 58.2, we'll use this notion to determine the homotopy type of X.

58.2. a) $X = B^2 \times S^1$:



Define a deformation retraction $H: X \times I \to X$ of X onto S^1 as follows:

For $(x,y) \in B^2 \times S^1 = X$, let
$$H((x,y),t) = ((1-t)x, y).$$

Since $t \in [0,1]$ we have $\|(1-t)x\| \leq \|x\| \leq 1$ whenever $x \in B^2$, hence $H((x,y),t) \in X$.

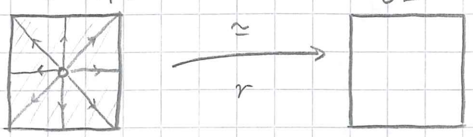
Moreover,
- $H((x,y),0) = (x,y)$,
- $H((x,y),1) = (0,y) \in S^1$
- $H((0,y),t) = (0,y)$.

Thus X deformation retracts onto S^1.

Choosing your favorite base point y_0 of S^1, we can use Theorem 58.3 to conclude that the inclusion $j: (S^1, y_0) \hookrightarrow (X, y_0)$ induces an isomorphism $j_*: \pi_1(S^1, y_0) \xrightarrow{\cong} \pi_1(X, y_0)$, i.e., $\pi_1(B^2 \times S^1, y_0) \cong \mathbb{Z}$.

b) $X = T^2 \setminus \{pt\}$ Claim X deformation retracts onto the figure 8-space $S^1 \vee S^1$.

Let $q : I^2 \longrightarrow T^2$ be the quotient map $q(x) := [x] \in T^2$.



$$\begin{tikzcd} I^2 \setminus \{pt\} \arrow[r, "\simeq", "r"'] \arrow[d, "q|_{I^2 \setminus \{pt\}}"'] \arrow[dr, phantom, "\circ"] & \partial I^2 \arrow[d, "q|_{\partial I^2}"] \\ T^2 \setminus \{pt\} \arrow[r, "\simeq", "\bar{r}"'] & S^1 \vee S^1 \end{tikzcd}$$

$\boxed{I^2 \setminus \{pt\} \text{ deformation retracts onto its boundary.}}$

May assume $pt \in \operatorname{int}(I^2)$.

Define a retraction $r : I^2 \setminus \{pt\} \longrightarrow \partial I^2$ by

$r(x) = \partial I^2 \cap L(pt, x)$, where $L(pt, x)$ is the line through pt and x.

Then $r \circ j = id_{\partial I^2}$, where $j : \partial I^2 \hookrightarrow I^2 \setminus \{pt\}$ is the inclusion,

and $j \circ r \simeq id_{I^2 \setminus \{pt\}}$ via straight line homotopy H,

hence ∂I^2 is a deformation retraction of $I^2 \setminus \{pt\}$.

Define a retraction $\bar{r} : T^2 \setminus \{pt\} \longrightarrow S^1 \vee S^1$

by $\bar{r}([x]) = [r(x)]$ as in the diagram above.

Then $\bar{r}([x]) = [r(x)] = [x]$ for any $[x] \in S^1 \vee S^1$, since

$S^1 \vee S^1$ is the image of ∂I^2 under q.

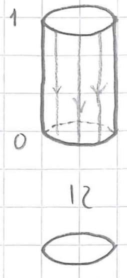
Similarly we define the homotopy $\bar{j} \circ \bar{r} \simeq id_{T^2 \setminus \{pt\}}$ by

using the one we have for $I^2 \setminus \{pt\}$:

$$\begin{tikzcd} I^2 \setminus \{pt\} \times I \arrow[r, "H"] \arrow[d, "q \times id_I"'] \arrow[dr, phantom, "\circ"] & I^2 \setminus \{pt\} \arrow[d, "q"] \\ T^2 \setminus \{pt\} \times I \arrow[r, "\bar{H}"'] & T^2 \setminus \{pt\} \end{tikzcd}$$

Hence $T^2 \setminus \{pt\}$ deformation retracts onto $S^1 \vee S^1$, yielding that the fundamental group of $T^2 \setminus \{pt\}$ is isomorphic to the fundamental group of the figure 8.

c) $X = S^1 \times I$



Define a deformation retraction of X onto S^1 by

$$H((x,y),t) = (x, (1-t)y) \quad \text{for any } (x,y) \in S^1 \times I = X.$$

This yields $\pi_1(X, x_0) \cong \mathbb{Z}$.

d) $X = S^1 \times \mathbb{R}$

Here we can use the same homotopy as in c),

$$H((x,y),t) = (x, (1-t)y),$$

to obtain that S^1 is a deformation retract of X.

Thus $\pi_1(X, x_0) \cong \mathbb{Z}$.

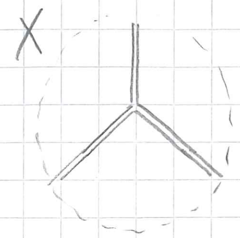
e) $X = \mathbb{R}^3 \setminus \{\text{nonnegative } x-, y- \text{ and } z\text{-axes}\} =: \mathbb{R}^3 \setminus T$

Consider X as a subspace of $\mathbb{R}^3 \setminus \{0\}$.

Recall from Theorem 58.2 that $\mathbb{R}^3 \setminus \{0\} \simeq S^2$ via a deformation retraction $j \circ r \simeq id_{\mathbb{R}^3 \setminus \{0\}}$, where $r : \mathbb{R}^3 \setminus \{0\} \to S^2$ is the retraction $x \longmapsto \frac{x}{\|x\|}$.

Under r, the set T is mapped onto $\{(1,0,0), (0,1,0), (0,0,1)\} \subseteq S^2$, hence $X \simeq S^2 \setminus \{(1,0,0), (0,1,0), (0,0,1)\}$.

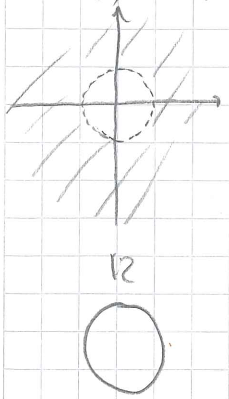
Now $S^2 \setminus \{pt\} \simeq \mathbb{R}^2$ by stereographic projection (see proof of Theorem 59.3), hence $X \simeq \mathbb{R}^2 \setminus \{p, q\} \simeq S^1 \vee S^1$, the figure 8-space (see Example 2 in §58).



$X \subseteq \mathbb{R}^3 \setminus \{0\} \simeq S^2$

$\simeq S^2 \setminus \{P_1, P_2, P_3\} \simeq \mathbb{R}^2 \setminus \{p, q\} \simeq S^1 \vee S^1.$

f) $X = \{ x \mid \|x\| > 1 \} \subseteq \mathbb{R}^2$

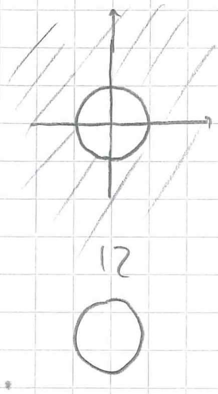


X deformation retracts onto the circle with radius (e.g.) 2; $X \simeq \{ x \in \mathbb{R}^2 : \|x\| = 2 \}$ via

$$H(x,t) = (1-t)x + 2t \frac{x}{\|x\|}.$$

Hence $\pi_1(X, x_0) \cong \mathbb{Z}$.

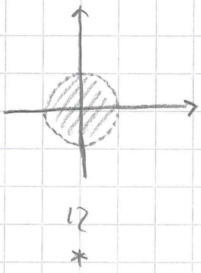
g) $X = \{ x \mid \|x\| \geq 1 \} \subseteq \mathbb{R}^2$



$X \simeq S^1$ via $H(x,t) = (1-t)x + t \frac{x}{\|x\|}$

$\Rightarrow \pi_1(X, x_0) \cong \mathbb{Z}$.

h) $X = \{ x \mid \|x\| < 1 \} \subseteq \mathbb{R}^2$

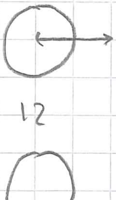


$X = \operatorname{int} B^2$, the open unit disk, which is contractible via

$$H(x,t) = (1-t)x.$$

$\Rightarrow \pi_1(X, x_0) = 0$.

i) $X = S^1 \cup (\mathbb{R}_+ \times 0)$



$X \simeq S^1$ by

$$H(x,t) = (1-t)x + t \frac{x}{\|x\|}.$$

$\Rightarrow \pi_1(X, x_0) \cong \mathbb{Z}$.

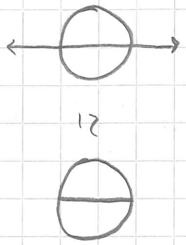
j) $X = S^1 \cup (\mathbb{R}_+ \times \mathbb{R})$

$X \simeq S^1$ by

$H(x,t) = (1-t)x + t\frac{x}{\|x\|}$

$\Rightarrow \pi_1(X, x_0) \cong \mathbb{Z}$.

k) $X = S^1 \cup (\mathbb{R} \times 0)$

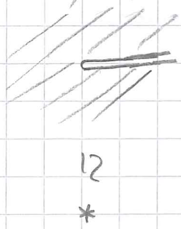


X deformation retracts onto the θ-space of Example 3 by contracting the two lines outside the circle.

Explicitly: $H(x,t) = \begin{cases} x, & \|x\| \leq 1 \\ (1-t)x + t\frac{x}{\|x\|}, & \|x\| > 1. \end{cases}$

Thus $\pi_1(X, x_0)$ is the fundamental group of the figure 8-space.

l) $X = \mathbb{R}^2 \setminus (\mathbb{R}_+ \times 0)$



X is contractible by e.g.,

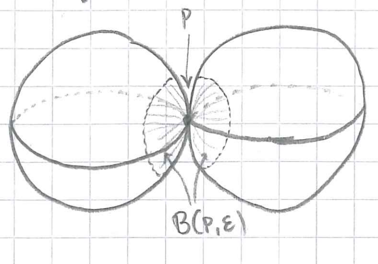
$H(x,t) = (1-t)x + t \cdot (-1, 0)$

Thus $\pi_1(X, x_0) = 0$.

59.1. $X = S^2 \vee S^2$, union of two copies of S^2 with one point P in common. What is $\pi_1(X,P)$?

- Claim $\pi_1(X,P)$ is trivial.

Idea: Find a suitable open covering of X and apply Corollary 59.2.



Let $U = S^1 \vee B(P,\varepsilon)$, that is, U is the first sphere along with a small open ball in the second sphere, with center the common point P. Similarly, let $V = B(P,\varepsilon) \vee S^1$ be the second sphere along with an open ball in the first sphere.

Then $X = U \cup V$ is an open covering, and

$U \cap V = B(P,\varepsilon) \vee B(P,\varepsilon)$ is the two balls with one point P in common. As $B(P,\varepsilon)$ is contractible (it deformation retracts onto P by straight line homotopy), Corollary 59.2 gives that X is simply connected, i.e., $\pi_1(X,P) = 0$.

59.2. The problem with the "proof" is that it is not necessarily possible to choose a point p of S^2 not lying in the image of f.

Examples are provided by the so-called space filling curves; see e.g. the Peano map (§44, Thm 44.1) for an example of a continuous surjective map $f : I \to I^2$.

60.1. • $\pi_1(S^1 \times B^2)$ was computed in exercise 59.1. a), the result being $\pi_1(S^1 \times B^2) \cong \mathbb{Z}$.

• For $S^1 \times S^2$, we use Theorem 60.1 to deduce

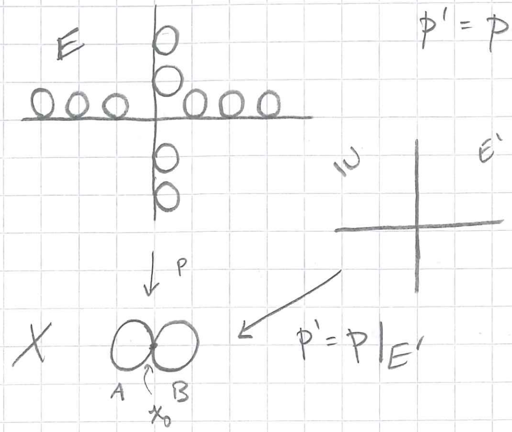
$$\pi_1(S^1 \times S^2, x_0 \times y_0) \cong \pi_1(S^1, x_0) \times \pi_1(S^2, y_0)$$
$$\cong \mathbb{Z} \times \{0\} \cong \mathbb{Z},$$

since S^n is simply connected for $n > 1$.

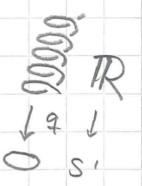
60.2. Show that $B^2 / \underset{\forall x \in S^1 = \partial B^2}{x \sim -x}$ is homeomorphic to P^2.

- Let $\sim$ denote the given equivalence relation on B^2.
Write the elements of $B^2/\sim$ as $[b]$, and the elements of P^2 as $\{x, -x\}$.
Consider B^2 as the closed upper hemisphere S^2_+ of S^2.
Let $p: S^2 \to P^2$ be the quotient map, $p(x) = \{x, -x\}$, and let f be the restriction $f = p|_{S^2_+} : S^2_+ = B^2 \to P^2$.
If $x \in \partial B^2 = S^1$, $-x \in \partial B^2$ also and $f(x) = f(-x) = \{x, -x\}$.
Hence f induces a map $\bar{f} : B^2/\sim \to P^2$, which is continuous since quotient maps, restriction and induced map on quotients are all continuous.
Moreover, $\bar{f}$ is bijective. Indeed, f is injective on the interior of B^2, and if $x \in \partial B^2$, then $[x] = [-x]$ in $B^2/\sim$.
Hence $\bar{f}$ is injective. Surjectivity follows from surjectivity of f.
Now recall Theorem 26.6: If $\varphi: X \to Y$ is a continuous bijection with X compact and Y Hausdorff, then φ is a homeomorphism.
A quotient of a compact space is compact, so $B^2/\sim$ is compact. Since P^2 is Hausdorff, the claim follows.

60.3. $p : E \to X$ covering map from proof of Lemma 60.5, $E' \subseteq E$ union of x- and y-axis, $p' = p|_{E'}$. Show that p' is not a covering map. (X = figure 8-space)

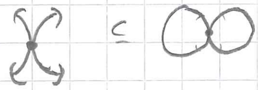


Note: $p'|_{(x\text{-axis})}$ is the standard covering map $\mathbb{R} \xrightarrow{q} S^1$, $x \mapsto (\cos 2\pi x, \sin 2\pi x)$ of the circle A. Similarly $p'|_{(y\text{-axis})}$ is standard covering of the second circle B.

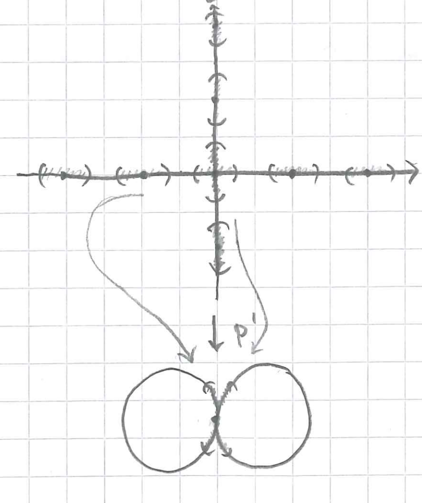


Thus "patching these covering maps together" doesn't yield a covering map.

To show that p' is not a covering map, consider the common point x_0 of the two circles A, B constituting the space X. Any open neighborhood U around x_0 looks like



, i.e., two open intervals meeting at one point.



$(p')^{-1}(U)$ consists of small open intervals centered at $(n, 0)$ and $(0, n)$ for each $n \in \mathbb{Z}$. But U is not homeomorphic to an interval.

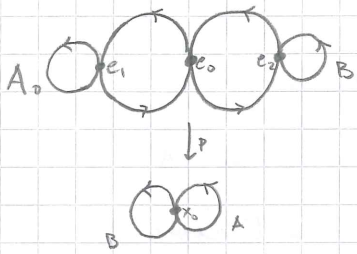
Indeed, a continuous map $U \to I$ cannot be injective.

60.4. The space P^1 and the covering map $p: S^1 \to P^1$ are familiar ones. What are they?

- the space $P^1 = S^1/_{x\sim -x}$ is homeomorphic to a half-circle with the two end points identified, i.e., S^1.

Under this homeomorphism, the quotient map $p: S^1 \to P^1$ corresponds to the covering map $z \mapsto z^2 : S^1 \to S^1$ of Example 3 in §53.

60.5. Use the following covering space to show $\pi_1(\text{figure } 8)$ is not abelian:



Here p wraps A_1 around A twice,
—"— B_1 —"— B —"—,
and p is homeomorphism from A_0 (resp. B_0) onto A (resp. B).

- By Theorem 54.3, enough to find loops f, g such that the lifts $\widetilde{f \cdot g}$ and $\widetilde{g \cdot f}$ have different end points.

Let f be the loop that travels once around A counterclockwise, and g loop once around B counterclockwise. Then the lifts $\widetilde{f \cdot g}$, $\widetilde{g \cdot f}$ are as follows: