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Elements of Mathematics

General Topology

Part 1

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INTRODUCTION

Most branches of mathematics involve structures of a type different from the *algebraic* structures (groups, rings, fields, etc.) which are the subject of the book Algebra of this series: namely structures which give a mathematical content to the intuitive notions of *limit*, *continuity* and *neighbourhood*. These structures are the subject matter of the present book.

Historically, the ideas of limit and continuity appeared very early in mathematics, notably in geometry, and their role has steadily increased with the development of analysis and its applications to the experimental sciences, since these ideas are closely related to those of *experimental determination* and *approximation*. But since most experimental determinations are *measurements*, that is to say determinations of one or more *numbers*, it is hardly surprising that the notions of limit and continuity in mathematics were featured at first only in the theory of real numbers and its outgrowths and fields of application (complex numbers, real or complex functions of real or complex variables, Euclidean geometry and related geometries).

In recent times it has been realized that the domain of applicability of these ideas far exceeds the real and complex numbers of classical analysis (see the Historical Note to Chapter I). Their essential content has been extracted by an effort of analysis and abstraction, and the result is a tool whose usefulness has become apparent in many branches of mathematics.

In order to bring out what is essential in the ideas of limit, continuity and neighbourhood, we shall begin by analysing the notion of *neighbourhood* (although historically it appeared later than the other two). If we start from the physical concept of approximation, it is natural to say that a subset A of a set E is a neighbourhood of an element a of A if, whenever we replace a by an element that "approximates" a, this new element will also belong to A, provided of course that the "error" involved is small enough; or, in other words, if all the points of E which are "sufficiently near" a belong to A. This definition is meaningful whenever precision can be given to the concept of sufficiently small error or of an element sufficiently near another. In this direction, the first idea was to suppose that the "distance" between two elements can be measured by a (positive) real number. Once the "distance" between any two elements of a set has been defined, it is clear how the "neighbourhoods" of an element a should be defined : a subset will be a neighbourhood of a if it contains all elements whose distance from a is less than some preassigned strictly positive number. Of course, we cannot expect to develop an interesting theory from this definition unless we impose certain conditions or axioms on the "distance" (for example, the inequalities relating the distances between the three vertices of a triangle which hold in Euclidean geometry should continue to hold for our generalized distance). In this way we arrive at a vast generalization of Euclidean geometry. It is convenient to continue to use the language of geometry : thus the elements or a set on which a "distance" has been defined are called *points*, and the set itself is called a *space*. We shall study such spaces in Chapter IX.

So far we have not succeeded in freeing ourselves from the real numbers. Nevertheless, the spaces so defined have a great many properties which can be stated without reference to the "distance" which gave rise to them. For example, every subset which contains a neighbourhood of a is again a neighbourhood of a, and the intersection of two neighbourhoods of a is a neighbourhood of a. These properties and others have a multitude of consequences which can be deduced without any further recourse to the "distance" which originally enabled us to define neighbourhoods. We obtain statements in which there is no mention of magnitude or distance.

We are thus led at last to the general concept of a topological space, which does not depend on any preliminary theory of the real numbers. We shall say that a set E carries a *topological structure* whenever we have associated with each element of E, by some means or other, a family of subsets of E which are called *neighbourhoods* of this element — provided of course that these neighbourhoods satisfy certain conditions (the *axioms* of topological structures). Evidently the choice of axioms to be imposed is to some extent arbitrary, and historically has been the subject of a great deal of experiment (see the Historical Note to Chapter I). The system of axioms finally arrived at is broad enough for the present needs of mathematics, without falling into excessive and pointless generality.

A set carrying a topological structure is called a *topological space* and its elements are called *points*. The branch of mathematics which studies topological structures bears the name of *Topology* (etymologically, "science of place", not a particularly expressive name), which is preferred nowadays to the earlier (and synonymous) name of *Analysis situs*.

To formulate the idea of neighbourhood we started from the vague concept of an element "sufficiently near" another element. Conversely, a topological structure now enables us to give precise meaning to the phrase "such and such a property holds for all points *sufficiently near* a": by definition this means that the set of points which have this property is a neighbourhood of a for the topological structure in question.

From the notion of neighbourhood there flows a series of other notions whose study is proper to topology: the interior of a set, the closure of a set, the frontier of a set, open sets, closed sets, and so on (see Chapter I, § 1). For example, a subset A is an *open* set if, whenever a point a belongs to A, all the points sufficiently near a belong to A; in other words, if A is a neighbourhood of each of its points. The axioms for neighbourhoods have certain consequences for all these notions; for example, the intersection of two open sets is an open set (because we have supposed that the intersection of two neighbourhoods of a is a neighbourhood of a). Conversely, we can start from one of these derived notions instead of starting from the notion of a neighbourhood; for example, we may suppose that the open sets are known, and take as axioms the properties of the family of open sets (one of these properties has just been stated, by way of example). We can then verify that, from knowledge of the open sets, the neighbourhoods can be reconstructed; the axioms for neighbourhoods are now consequences of the new axioms for open sets that we took as a starting point. Thus a topological structure can be defined in various different ways which are basically equivalent. In this book we shall start from the notion of *open set*, because the corresponding axioms are the simplest.

Once topological structures have been defined, it is easy to make precise the idea of *continuity*. Intuitively, a function is continuous at a point if its value varies as little as we please whenever the argument remains sufficiently near the point in question. Thus continuity will have an exact meaning whenever the space of arguments and the space of values of the function are topological spaces. The precise definition is given in Chapter I, § 2.

As with continuity, the idea of a *limit* involves two sets, each endowed with suitable structures, and a mapping of one set into the other. For example, the limit of a sequence of real numbers a_n involves the set $\mathbf{N}$ of natural numbers, the set $\mathbf{R}$ of real numbers, and a mapping of the former set into the latter. A real number a is then said to be a limit of the sequence if, whatever neighbourhood V of a we take, this neighbourhood contains all the a_n except for a finite number of values of n; that is, if the set of natural numbers n for which a_n belongs to V is a subset of $\mathbf{N}$ whose complement is finite. Note that $\mathbf{R}$ is assumed to carry a topological structure, since we are speaking of neigbourhoods; as to the set $\mathbf{N}$, we have made a certain family of subsets play a particular part, namely those subsets whose complement is finite. This is a general fact: whenever we speak of limit, we are considering a mapping f of a set E into a topological space F, and we say that f has a point a of F as a limit if the set of elements x of E whose image $f(x)$ belongs to a neighbourhood V of a [this set is just the "inverse image" $\overset{-1}{f}(V)$] belongs, whatever the neighbourhood V, to a certain family $\mathfrak{F}$ of subsets of E, given beforehand. For the notion of limit to have the essential properties ordinarily attributed to it, the family $\mathfrak{F}$ must satisfy certain axioms, which are stated in Chapter I, § 6. Such a family $\mathfrak{F}$ of subsets of E is called a *filter* on E. The notion of a filter, which is thus inseparable from that of a limit, appears also in other contexts in topology; for example, the neighbourhoods of a point in a topological space form a filter.

The general study of all these notions is the essential purpose of Chapter I. In addition, particular classes of topological spaces are considered there, spaces which satisfy more restrictive axioms, or spaces obtained by particular procedures from other given spaces.

As we have already said, a topological structure on a set enables one to give an exact meaning to the phrase "whenever x is sufficiently near a, x has the property $P\{x\}$". But, apart from the situation in which a "distance" has been defined, it is not clear what meaning ought to be given to the phrase "every pair of points x, y which are sufficiently near each other has the property $P\{x, y\}$", since *a priori* we have no means of comparing the neighbourhoods of two different points. Now the notion of a pair of points near to each other arises frequently in classical analysis (for example, in propositions which involve uniform continuity). It is therefore important that we should be able to give a precise meaning to this notion in full generality, and we are thus led to define structures which are richer than topological structures, namely *uniform structures*. They are the subject of Chapter II.

The other chapters of this Book are devoted to questions in which, in addition to a topological or uniform structure, there is some other structure present. For example a *group* which carries a suitable topology (compatible in a certain sense with the group structure) is called a *topological group*. Topological groups are studied in Chapter III, and we shall see there in particular how every topological group can be endowed with certain uniform structures.

In Chapter IV we apply the preceding principles to the field of rational numbers. This enables us to define the field of real numbers; because of its importance, we study it in considerable detail. In the succeeding chapters, starting from the real numbers, we define certain topological spaces which are of particular interest in applications of topology to classical geometry: finite-dimensional vector spaces, spheres, projective spaces, etc. We consider also certain topological groups closely related to the additive group of real numbers, which we characterize axiomatically, and this leads us to the definition and elementary properties of the most important functions of classical analysis: the exponential, logarithmic and trigonometric functions.

In Chapter IX we revert to general topological spaces, but now with a new instrument, namely the real numbers, at our disposal. In particular we study spaces whose topology is defined by means of a "distance"; these spaces have properties, some of which cannot be extended to more general spaces. In Chapter X we study sets of mappings of a topological space into a uniform space (function spaces); these sets, suitably topologized, have interesting properties which already play an important part in classical analysis.

HISTORICAL NOTE

(Numbers in brackets refer to the bibliography at the end of this note.)

The ideas of limit and continuity go back to antiquity, and a complete history of them could not be written without studying systematically from this point of view not only the Greek mathematicians but also the Greek philosophers and Aristotle in particular. It would also be necessary to trace the evolution of these ideas through Renaissance mathematics and the beginnings of the differential and integral calculus. Such a study, though it would undoubtedly be interesting to undertake, would go far beyond the framework of this note.

It is Riemann who should be considered as the creator of topology, as of so many other branches of modern mathematics. He was the first to attempt to formulate the notion of a topological space; he conceived the idea of an autonomous theory of such spaces; he defined invariants (the "Betti numbers") which were to play a pre-eminent part in the later development of topology; and he was the first to apply topology to analysis (periods of abelian integrals). But the current of ideas in the first half of the nineteenth century had prepared the path for Riemann in more ways than one. In the first place, the desire to put mathematics on a firm basis, which was the cause of so many important researches throughout the nineteenth century and up to the present day, had led to a correct understanding of the notions of a convergent series and a sequence of numbers tending to a limit (Cauchy, Abel) and to the notion of a continuous function (Bolzano, Cauchy). On the other hand, the geometrical representation (by points of a plane) of the complex numbers (or, as they had hitherto been called, "imaginary" or even "impossible" numbers) which was due to Gauss and Argand, had become familiar to the majority of mathematicians; it constituted an advance of the same order as the adoption, in our century, of the language of geometry in the study of Hilbert space, and contained the germ of the possibility of a geometrical representation of every object capable of continuous variation. Gauss, who was in any case naturally led to such concepts by his researches on the foundations of geometry, on non-Euclidean geometry and on curved surfaces, seems to have had this possibility already in mind, for he uses the words "magnitude twice extended" when defining (independently of Argand and the French mathematicians) the geometrical representation of complex numbers ([1], pp. 101-103 and 175-178).

Riemann's work on algebraic functions and their integrals and his reflections on the foundations of geometry (largely inspired by his study of Gauss's work) led him to formulate a program of study, which is precisely that of modern topology, and to begin to realize this program. Here, for example, is what he says in his theory of abelian functions ([2], p. 91):

"*In the study of functions obtained by integrating exact differentials, some theorems of Analysis situs are almost indispensable. By this name, which was used by Leibnitz, although perhaps in a somewhat different sense, should be called that part of the theory of continuous magnitudes which studies these magnitudes, not independently of their position and by measuring them in terms of each other, but rather by abstracting all ideas of measurement and considering only their relations of position and inclusion. I reserve to a later occasion an investigation completely independent of all measurement...*"

And in his famous inaugural lecture "On the hypotheses which underlie Geometry" ([2], p. 272):

"... *the general concept of a magnitude many times extended* (*) *which contains as a particular case that of spatial magnitude, has remained completely unexplored...*" (p. 272)

"... *The notion of magnitude presupposes that an element is capable of different determinations. According as one can pass from one determination to another by a continuous process of transition or not, these determinations form a continuous or a discrete manifold: in the former case the determinations are called points of the manifold...*" (p. 273)

"... *Measurement consists of superposition of the magnitudes to be compared, hence in order to measure we need some means of using one magnitude as a yardstick for another. In the absence of this we can compare two magnitudes only if one is part of the other... The investigations which can be undertaken in this context form a part of the theory of magnitudes which is independent of the theory of measurement and in which the magnitudes are considered not as existing independently of their position nor as expressible in terms of a unit of measurement, but as regions in a manifold. Such investigations have become necessary in several parts of mathematics, in particular in the theory of many-valued analytic functions...* " (p. 274)

(*) As the sequel shows, Riemann means by this phrase a subset of a topological space of arbitrary dimension.

"... *The determination of position in a given manifold, whenever this is possible, can be reduced to a finite number of numerical determinations. There are however manifolds in which the determination of position requires not a finite number but an infinite sequence or even a continuous manifold of determinations of magnitudes. For example, the possible determinations of a function on a given domain, or the possible forms of a spatial figure, give manifolds of this type.*" (p. 276)

Note in this last phrase the first idea of a study of functional spaces; Riemann had already expressed the same idea in his dissertation: "*the totality of these functions*", he stated in connection with the minimal problem known as Dirichlet's principle, "*forms a connected domain which is closed in itself*" ([2], p. 30); this, though imperfectly expressed, is nevertheless the germ of the proof which Hilbert was later to give of Dirichlet's principle, and of most of the applications of function spaces to the calculus of variations.

As we have said, Riemann began the execution of this grandiose program by defining the "Betti numbers", first for a surface ([2], pp. 92-93) and later ([2], pp. 479-482; cf. also [3]) for a manifold of any dimension, and applied this definition to the theory of integrals; for this and for the considerable development of this theory since Riemann's time we refer the reader to the Historical Notes to the chapters on algebraic topology in this series of volumes.

Before a general theory of topological spaces, such as Riemann had envisaged, could be developed, it was necessary that the theory of real numbers, of sets of numbers, of sets of points on a line, in a plane and in space should be more systematically investigated than they had been in Riemann's time. Such investigations were related on the other hand to research into the nature of irrational numbers (semi-philosophical by Bolzano and essentially mathematical by Dedekind) and to progress in the theory of functions of a real variable in which Riemann himself made an important contribution by his definition of the integral and his theory of trigonometrical series, and to which du Bois-Reymond, Dini and Weierstrass, among others, contributed; they were the work of the second half of the nineteenth century, and especially the work of Cantor, who was the first to define (originally on the line, later in Euclidean space of n dimensions) the notions of point of accumulation, closed set, open set, perfect set, and obtained the essential results on the structure of these sets on the line (cf. the Historical Note to Chapter IV). In this context, not only the works of Cantor [4] should be consulted, but also his extremely interesting correspondence with Dedekind [5], where the idea of dimensionality as a topological invariant can be found clearly expressed. The later progress of the theory is traced in a semi-historical, semi-systematic form in Schœnflies' book [6]; by far the most important acquisition was the theorem of Borel-Lebesgue, namely the fact that every bounded closed subset of Euclidean n-space $\mathbf{R}^n$ (cf. Chapter VI, § 1) satisfies axiom (C''') of § 9 of this chapter (the theorem was first proved by Borel for a closed interval on the line and a countable family of open intervals covering it).

Cantor's ideas had originally met with vigorous opposition (cf. the Historical Note to Book I, Chapters I-IV). At any rate his theory of point-sets on the line and in the plane was quickly made use of and disseminated by the French and German schools of function-theory (Jordan, Poincaré, Klein, Mittag-Leffler, and later Hadamard, Bore, Baire, Lebesgue, etc.); each of the early Borel treatises, in particular, contains an elementary exposition of this theory (see for example [7]). As these ideas spread, their possible application to sets, not of points but of curves or functions, began to be considered in various quarters, as witness the title "On the limit curves of a variety of curves" of a memoir by Ascoli in 1883 [8] and a communication by Hadamard to the congress of mathematicians at Zürich in 1896 [9]; all this is closely related to the introduction of "line functions" by Volterra in 1887 and to the creation of "functional calculus" or theory of functions in which the argument is a function (cf. Voltera's book on functional analysis [10]). On the other hand, Hilbert's famous memoir [11], in which, taking up Riemann's ideas again, he proved the existence of the minimum in Dirichlet's principle and inaugurated the "direct method" in the calculus of variations, showed clearly the importance of considering sets of functions in which the Bolzano-Weierstrass principle holds, that is to say in which every sequence has a convergent subsequence. Such sets were beginning in any case to play an important part, not only in the calculus of variations but also in the theory of functions of a real variable (Ascoli, Arzelà) and a little later in the theory of functions of a complex variable (Vitali, Carathéodory, Montel). Finally the study of functional equations, and especially the solution by Fredholm of the type of equation which bears his name, made it commonplace to consider a function as an argument and a set of functions as a set of points, and as natural to use the language of geometry in this context as in Euclidean space of n dimensions (a space which equally eludes "intuition" and for this reason remained long an object of distrust to many mathematicians). In particular the memorable work of Hilbert on integral equations [12] led to the definition and geometrical study of Hilbert space by Erhard Schmidt [14], in complete analogy with Euclidean geometry.

Meanwhile the concept of an axiomatic theory had acquired more and more importance, thanks to much work on the foundations of geometry; here Hilbert's contributions [13] had a particularly decisive influence. In the course of this work, Hilbert had been led to formulate in 1902 ([13], p. 180) the first axiomatic definition of the "manifold twice extended" in the sense of Riemann, a definition which constituted, said Hilbert, "the foundation of a rigorous axiomatic treatment of *Analysis* *situs*". Hilbert also made use of neighbourhoods (in a sense restricted by the demands of the problem to which he limited himself).

The first attempts to abstract what is common to properties of sets of points and sets of functions are due to Fréchet [15] and F. Riesz [16]. The former started from the notion of countable limit and did not succeed in constructing a convenient and fruitful system of axioms, but at least he recognized the relationship between the principle of Bolzano-Weierstrass [which is just axiom (C) of § 9, restricted to countable sequences] and the Borel-Lebesgue theorem [axiom (C''') of § 9]; in this connection he introduced the word "compact", although in a sense somewhat different from that in which it is used in this series of volumes. As to F. Riesz, who took as his starting point the concept of point of accumulation (or rather of "derived set", which amounts to the same thing), his theory was again incomplete and appeared only in outline form.

General topology as it is understood today began with Hausdorff ([17], Chapters 7, 8, 9), who again took up the concept of neighbourhood (by which he meant what in the terminology of this series of volumes is called an "open neighbourhood") and chose from Hilbert's axioms for neighbourhoods in the plane those which gave his theory all the precision and generality desired. The axioms he took as a starting-point were essentially (taking into account the difference between his concept of neighbourhood and ours) axioms (V_I), (V_{II}), (V_{III}), (V_{IV}) of § 1 and (H) of § 8, and the chapter in which he develops the consequences of these axioms has remained a model of axiomatic theory, abstract but adapted in advance to applications. Hausdorff's work was naturally the point of departure for later research in general topology and especially for the work of the Moscow school, which was largely directed towards the problem of metrization (cf. the Historical Note to Chapter IX); here we recall especially the definition of compact spaces (under the name of "bicompact spaces") by Alexandroff and Urysohn, and Tychonoff's proof of the compactness of products of compact spaces [19]. Finally, the introduction of filters by H. Cartan [20] has brought to topology a valuable instrument, usable in all sorts of applications (in which it replaces to advantage the notion of "Moore-Smith convergence" [18]). Furthermore, the development of the theorem on ultrafilters (Theorem 1, § 6), has clarified and simplified the theory.