In this solution, the word 'map' means 'continuous function'.

Problem 1a

A topological space X is Hausdorff if for each pair of distinct points $p, q \in X$ there exist open, disjoint subsets $U, V \subset X$ with $p \in U$ and $q \in V$.

A topological space X is locally compact if for each point $p \in X$ there exist an open set $U \subset X$ and a compact set $C \subset X$ with $p \in U \subset C$.

Problem 1b

The open sets in $Y \cup \{\infty\}$ are of the form V with $V \subset Y$ open or $(Y - D) \cup \{\infty\}$ with $D \subset Y$ compact. In the first case

$$f_1^{-1}(V) = f^{-1}(V)$$

and $f^{-1}(V)$ is open in X since f is continuous, so $f_1^{-1}(V)$ is open in $X \cup \{\infty\}$. In the second case,

$$f_1^{-1}((Y - D) \cup \{\infty\}) = (X - f^{-1}(D)) \cup \{\infty\}$$

and $f^{-1}(D) \subset X$ is compact since f is proper, so $f_1^{-1}((Y - D) \cup \{\infty\})$ is open in $X \cup \{\infty\}$. This proves that f_1 is continuous.

Problem 1c

The open sets in $B \cup \{\infty\}$ are of the form W with $W \subset B$ open or $(B - E) \cup \{\infty\}$ with $E \subset B$ compact. In the first case,

$$i_2^{-1}(W) = W$$

is open in B, hence also in Y, since B is open in Y. Thus $i_2^{-1}(W)$ is open in $Y \cup \{\infty\}$. In the second case,

$$i_2^{-1}((B - E) \cup \{\infty\}) = (Y - E) \cup \{\infty\}$$

where $E \subset Y$ is compact. Thus $i_2^{-1}((B - E) \cup \{\infty\})$ is open in $Y \cup \{\infty\}$. This proves that i_2 is continuous.

Problem 1d

To check that g is proper, let $M \subset B$ be compact. Then

$$\begin{aligned} g^{-1}(M) &= \{x \in A \mid g(x) \in M\} \\ &= \{x \in X \mid f(x) \in B \text{ and } g(x) \in M\} \\ &= \{x \in X \mid f(x) \in M\} = f^{-1}(M)\,. \end{aligned}$$

Since M is compact and f is proper, we know that $f^{-1}(M)$ is compact. Hence $g^{-1}(M)$ is compact. This proves that g is proper.

Since B is open in Y and $f\colon X \to Y$ is continuous, the preimage $A = f^{-1}(B)$ is open in X. Hence $j\colon A \subset X$ is an open inclusion.

Problem 2a

If f is continuous, then the function

$$\begin{aligned} F\colon X &\longrightarrow X \times Z \\ x &\longmapsto (x, f(x)) \end{aligned}$$

is continuous, with image G. We know that the continuous image of a connected space is connected. Thus, if X is connected then G is connected.

Problem 2b

The restricted function $f|_{(0,1]}\colon (0,1] \to \mathbb{R}$ is continuous, so its graph

$$D = \{(x, \sin(1/x)) \mid 0 < x \leq 1\}$$

is connected. Furthermore, $(0,0)$ is in the closure of D in $[0,1] \times \mathbb{R} \subset \mathbb{R}^2$, since each ϵ-neighborhood of $(0,0)$ contains points of the form $(1/n\pi, 0) \in D$ for n sufficiently large. Hence $(0,0) \in \bar{D}$ and D is dense in G. We know that the closure of a connected subspace is connected. Applying this to the subspace D of G, we conclude that G is connected.

Problem 2c

The restricted function $f|_{\mathbb{R}^2 - \{(0,0)\}}$ is continuous, and $\mathbb{R}^2 - \{(0,0)\}$ is path connected. Its graph

$$E = \{(x, y, f(x, y) \mid (x, y) \neq (0,0)\}$$

is the continuous image of a path connected space, and is therefore path connected. It remains to show that the point $(0, 0, 0) = (0, 0, f(0, 0)) \in G$ belongs to the same path component as E.

To see this, use the path $p\colon [0, 1] \to G$ given by

$$p(t) = (t, 0, 0)\,.$$

It is continuous as a map to $\mathbb{R}^3$, and takes values in G, since $p(0) = (0, 0, 0)$ and $p(t) = (t, 0, f(t, 0))$ for $0 < t \le 1$, hence is continuous as a map to G in the subspace topology. Therefore $p(0) = (0, 0, 0)$ is in the same path component as $p(1) \in E$.

Problem 3a

Urysohn's lemma: If X is a normal space, and A and B are closed, disjoint subsets of X, then there exists a map $f\colon X \to [0, 1]$ such that $f(A) \subset \{0\}$ and $f(B) \subset \{1\}$.

Problem 3b

We assume that X is normal, and that A and B are closed, disjoint subsets of X. Let $f\colon X \to [0, 1]$ be a map such that $f(A) \subset \{0\}$ and $f(B) \subset \{1\}$.

Since $f\colon X \to [0, 1]$ and $g\colon X \to Z$ are continuous, so is $e = (f, g)\colon X \to [0, 1] \times Z$.

To check that e is injective, consider $x, y \in X$ with $e(x) = e(y)$. Then $f(x) = f(y)$ and $g(x) = g(y)$. If x and y both lie in $X - A$, then $x = y$ because $g|_{X-A}$ is injective. On the other hand, if x and y both lie in $X - B$, then $x = y$ because $g|_{X-B}$ is injective. We cannot have $x \in A$ and $y \in B$, because then $f(x) = 0$ is not equal to $f(y) = 1$. Finally, we cannot have $x \in B$ and $y \in A$, because then $f(x) = 1$ is not equal to $f(y) = 0$. This exhausts all possibilities, so $x = y$ and e is injective.

Problem 3c

To check that the continuous bijection $X \to e(X)$ given by e is open, consider any open subset $U \subset X$. We must prove that $e(U)$ is open in $e(X)$. Since $A \cap B = \emptyset$ we have $U = (U - A) \cup (U - B)$ and $e(U) = e(U - A) \cup e(U - B)$. Hence it is enough to show that $e(U - A)$ and $e(U - B)$ are open in $e(X)$. By symmetry it suffices to handle the case of $e(U - A)$. We do this by showing that $e(U - A)$ is open in $e(X - A)$, and that $e(X - A)$ is open in $e(X)$.

Let $h\colon e(X) \to g(X)$ be given by restricting the projection $\pi_2\colon [0,1]\times Z \to Z$ to the subspace $e(X)$. It restricts further to a map $h\colon e(X-A) \to g(X-A)$. The composition of $e\colon X-A \to e(X-A)$ and $h\colon e(X-A) \to g(X-A)$ is the homeomorphism $g\colon X-A \to g(X-A)$. It follows that $e\colon X-A \to e(X-A)$ is a bijective imbedding (= embedding), hence a homeomorphism. Because $U-A$ is open in $X-A$ we deduce that $e(U-A)$ is open in $e(X-A)$.

To show that $e(X-A)$ is open in $e(X)$ is equivalent to showing that its complement $e(A)$ is closed in $e(X)$. For this we use that $g\colon X-B \to g(X-B)$ is a homeomorphism. It follows as above that $e\colon X-B \to e(X-B)$ is a homeomorphism. Let

$$C = f^{-1}([0,1/2]) = \{x \in X \mid f(x) \le 1/2\}$$

be a (closed) subset of X, with $A \subset C \subset X - B$. Then A is closed in C, so $e(A)$ is closed in $e(C)$. Finally, $e(C) = e(X) \cap [0,1/2] \times Z$ is closed in $e(X)$, since $[0,1/2] \times Z$ is closed in $[0,1] \times Z$, so $e(A)$ is indeed closed in $e(X)$.