8th November, 2020

MAT3500/4500 - Fall 2020

Mandatory assignment 1 of 1

Submission deadline

Thursday 15th October 2020, 14:30 in Canvas.

Instructions

Students taking the course MAT4500 must submit the assignment typed in LaTeX. Students taking the course MAT3500 may choose between scanning handwritten notes or typing the solution directly on a computer (for instance with LaTeX). The latter is preferred. The assignment must be submitted as a single PDF file. Scanned pages must be clearly legible. The submission must contain your name, course and assignment number.

It is expected that you give a clear presentation with all necessary explanations. Remember to include all relevant plots and figures. Students who fail the assignment, but have made a genuine effort at solving the exercises, are given a second attempt at revising their answers. All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. If we doubt that you have understood the content you have handed in, we may request that you give an oral account.

In exercises where you are asked to write a computer program, you need to hand in the code along with the rest of the assignment. It is important that the submitted program contains a trial run, so that it is easy to see the result of the code.

Application for postponed delivery

If you need to apply for a postponement of the submission deadline due to illness or other reasons, you have to contact the Student Administration at the Department of Mathematics (e-mail: studieinfo@math.uio.no) well before the deadline.

All mandatory assignments in this course must be approved in the same semester, before you are allowed to take the final examination.

Complete guidelines about delivery of mandatory assignments:

uio.no/english/studies/admin/compulsory-activities/mn-math-mandatory.html

GOOD LUCK!

Problem 1. Let $X = (-1,1) \times (-1,1) \subset \mathbb{R}^2$. We let $\mathcal{F}$ be the family of open subsets W in $\mathbb{R}^2$ (with the usual topology) such that $(-1,1) \times \{1\} \subset W$ or $W = \emptyset$. Let $\mathcal{T} = \{W \cap X : W \in \mathcal{F}\}$.

(a) Prove that $\mathcal{T}$ is a topology on X.

(b) Prove that $(X, \mathcal{T})$ is a T_1-space. Is $(X, \mathcal{T})$ a Hausdorff space?

(c) What are the continuous mappings from $(X, \mathcal{T})$ into $\mathbb{R}$ (equipped with the usual topology)?

Solution:

(a) (i) $\emptyset \in \mathcal{F}$ and $\emptyset \cap X = \emptyset$, so $\emptyset \in \mathcal{T}$. Further, $\mathbb{R}^2 \in \mathcal{F}$ and $\mathbb{R}^2 \cap X = X$, so $X \in \mathcal{T}$.

(ii) Suppose $\{U_\alpha\}_{\alpha \in \mathcal{A}} \subset \mathcal{T}$. Then $U_\alpha = X \cap W_\alpha$ for all α with $W_\alpha \in \mathcal{F}$. Set $W = \cup_\alpha W_\alpha$ and note that $W \in \mathcal{F}$. Then

$$U = \bigcup_\alpha U_\alpha = \bigcup_\alpha X \cap W_\alpha = X \cap \bigcup_\alpha W_\alpha = X \cap W,$$

so $U \in \mathcal{T}$.

(iii) Suppose $U_j \in \mathcal{T}$ for $j = 1, ..., n$. Then $U_j = X \cap W_j$ with $W_j \in \mathcal{F}$ for $j = 1, ..., n$. Set $W = \cap_{j=1}^n W_j$ and note that $W \in \mathcal{F}$ since the intersection is finite. Then

$$U = \bigcap_{j=1}^n U_j = \bigcap_{j=1}^n X \cap W_\alpha = X \cap \bigcap_{j=1}^n W_j = X \cap W,$$

so $U \in \mathcal{T}$.

(b) Let $p = (x_1, y_1)$ and $q = (x_2, y_2)$ be points in X. Choose $\epsilon > 0$ such that $q \notin B_\epsilon(p)$. Set $W = \{(x,y) \in \mathbb{R}^2 : y > y_2\} \cup B_\epsilon(p)$. Then $W \in \mathcal{F}$, and we have $p \in W \cap X$ while $q \notin W \cap X$. This shows that $(X, \mathcal{T})$ is a T_1 space.

On the other hand, X is not a Hausdorff space. This is because for any two nonempty sets $W_1, W_2 \in \mathcal{F}$ we have that $W_1 \cap W_2 \cap X \neq \emptyset$, and so $(X \cap W_1) \cap (X \cap W_2) = X \cap (W_1 \cap W_2)$ is nonempty.

(c) The only continuous functions are the constant functions. For if $f : X \to \mathbb{R}$ were continuous with $f(p) < f(q)$ for $p, q \in X$ and $p \neq q$, we could chose $f(p) < a < f(q)$ and get that

$$U = \{x \in X : f(x) < a\} \text{ and } V = \{x \in X : f(x) > a\}$$

were two disjoint open sets in $(X, \mathcal{T})$, contradicting the conclusion in (b).

Problem 2. Let X be a topological space and let $A \subset X$. We say that A is dense in X if $\overline{A} = X$ (recall that $\overline{A}$ is the intersection of all closed sets in X that contain A).

(a) Prove that $A \subset X$ is dense in X if and only if for any open subset $U \neq \emptyset$ of X we have that $U \cap A \neq \emptyset$.

(b) Prove that if $A \subset B$ is dense in B, and if $B \subset X$ is dense in X, then $A \subset X$ is dense in X (A and B are equipped with the subset topology).

(c) Prove that if $f : X \to Y$ is a continuous surjective map, and if $A \subset X$ is dense, then $f(A)$ is dense in Y.

(d) Suppose that $f : X \to Y$ is a continuous surjective map, and let $B \subset Y$ be dense. Is $f^{-1}(B)$ necessarily dense?

(e) Let $U, V \subset X$ be two open dense subsets. Prove that $U \cap V$ is open and dense.

Solution:

(a) Assume that A is dense in X. Then if $U \subset X$ is open and disjoint from A we have that $\bar{A} \subset X \setminus U$. Since $\bar{A} = X$ it follows that $U = \emptyset$, so any nonempty U intersects A.

Assume that any nonempty open set U intersects A. Let C be a closed set containing A. Then $X \setminus C$ is an open set disjoint from A, and it follows that $X \setminus C = \emptyset$, and hence that $C = X$. So $\bar{A} = X$.

(b) Let C be a closed set in X containing A. Then $C \cap B$ is closed in B, so $C \cap B = B$ since $\bar{A} = B$. So C is a closed set in X containing B, and since $\bar{B} = X$ we must have $C = X$. So $\bar{A} = X$.

(c) Let $V \subset Y$ be a nonempty open set. Since f is surjective, we have that $U = f^{-1}(V)$ is a nonempty open set. So there is a point $a \in A \cap U$. So $f(a) \in V$.

(d) No. Define $f : \mathbb{R} \to [0, \infty)$ by setting $f(x) = 0$ for $x \leq 0$ and $f(x) = x$ for $x \geq 0$. Then f is continuous, $(0, \infty)$ is dense in $[0, \infty)$, but $f^{-1}(0, \infty) = (0, \infty)$ which is not dense in $\mathbb{R}$.

(e) Let W be a nonempty open set. Then $U \cap W$ is nonempty and open since U is open and dense. Then $V \cap (U \cap W)$ is nonempty and open since V is open and dense. So $W \cap (U \cap V)$ is nonempty. Since W is arbitrary we conclude that $U \cap V$ is dense, and being the intersection of two open sets it is also open.

Problem 3. Let X be the set of continuous functions $f : (0, 1) \to \mathbb{R}$. For each $n \in \mathbb{Z}_+$ we set $K_n = [1/(n+1), 1 - 1/(n+1)]$. For each $n \in \mathbb{Z}_+$ and $f, g \in X$ set

$$d_n(f, g) = \max_{x \in K_n} \{|f(x) - g(x)|\}.$$

(a) Prove that for $f, g, h \in X$ and every $n \in \mathbb{Z}_+$ we have that

$$d_n(f, g) \leq d_n(f, h) + d_n(h, g).$$

Is d_n a metric on X for any $n \in \mathbb{Z}_+$?

(b) Let $\mathcal{B}$ consist of the following subsets $B_{f,n,r}$ of X, with $f \in X, n \in \mathbb{Z}_+$ and $r > 0$:
$$B_{f,n,r} = \{g \in X : d_n(g,f) < r\}.$$
Prove that $\mathcal{B}$ is a basis for a topology on X (this topology is called the *compact-open topology*).

Solution:

(a) For $f, g \in X$, and for a fixed $n \in \mathbb{Z}_+$, there exists a point $x_0 \in K_n$ such that $d_n(f,g) = |f(x_0) - g(x_0)|$. For $h \in X$ we have that
$$\begin{aligned} |f(x_0) - g(x_0)| &= |f(x_0) - h(x_0) + h(x_0) - g(x_0)| \\ &\leq |f(x_0) - h(x_0)| + |h(x_0) - g(x_0)| \\ &\leq d_n(f,h) + d_n(h,g) \end{aligned}$$
where we in the passage to the second line used the triangle inequality on $\mathbb{R}$, and in the passage to the third used that d_n is defined as a maximum.

We clearly have that $d_n(f,g) \geq 0$ for all $f, g \in X$ and that $d_n(f,g) = d_n(g,f)$. But for any $n \in \mathbb{Z}_+$ there are functions $f, g \in X$ such that $f(x) = g(x)$ for all $x \in K_n$ but $f \neq g$. Then $d_n(f,g) = 0$, but $f \neq g$. So d_n is not a metric on X. However, if we let X_n denote the space $C(K_n, \mathbb{R})$ of real valued continuous functions on K_n, we see that $d'_n = d_n|_{X_n}$ is a metric on X_n.

(b) Consider B_{f_1,n_1,r_1} and B_{f_2,n_2,r_2} with nonemtpty intersection containing $h \in X$. Set $s_j = d_{n_j}(h, f_j)$ for $j = 1, 2$. Set $\epsilon_j = r_j - s_j > 0$. Then
$$d_{n_j}(g,h) < \epsilon_j \Rightarrow d_{n_j}(g, f_j) \leq d_{n_j}(f_j, h) + d_{n_j}(h, g) < s_j + \epsilon_j = r_j,$$
which shows that $B_{h,n_j,\epsilon_j} \subset B_{f_j,n_j,r_j}$ for $j = 1, 2$. Without loss of generality we assume that $n_2 \geq n_1$, and note that $B_{h,n_2,\epsilon_1} \subset B_{h,n_1,\epsilon_1}$. So if we let $\epsilon = \min\{\epsilon_1, \epsilon_2\}$ we see that
$$B_{h,n_2,\epsilon} \subset B_{f_1,n_1,r_1} \cap B_{f_2,n_2,r_2}.$$

Problem 4. (Optional) Prove that the topology generated by the basis $\mathcal{B}$ in Problem 3 is metrizable.

Solution: For each $n \in \mathbb{Z}_+$ we set $\bar{d}_n = \min\{d_n, 1\}$. We have seen (when we metrized the product topology on $\mathbb{R}^\omega$) that $\bar{d}_n$ satisfies all the properties of being the metric, except for the one that failed in (a) (and it fails here for the same reason). If we set
$$d(f,g) = \max_{n \in \mathbb{Z}_+}\{\bar{d}_n(f,g)/n\},$$
Now $d_n(f,g) = 0$ if and only if $f|_{K_n} = g|_{K_n}$ for all n which is equivalent to $f = g$, and the only remaining axiom for being a metric which is not clear is perhaps the triangle inequality. But for any $f, g \in X$ there exists an $n \in \mathbb{Z}_+$ such that $d(f,g) = \bar{d}_n(f,g)/n$. Then for any $h \in X$ we have
$$\bar{d}_n(f,g)/n \leq \bar{d}_n(f,h)/n + \bar{d}_n(h,g)/n \leq d(f,h) + d(h,g),$$
where in the first inequality we used the observation above, and in the second we used that d is defined as a maximum.