SOME PROBLEMS ON METRIC SPACES AND SET THEORY/LOGIC

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1. Problems - metric spaces

Recall that for a metric space (X, d), the open ball of radius $r > 0$, centered at a point $x \in X$, is defined as

$$B_r(x) = \{y \in X : d(y, x) < r\}.$$

Recall further that a set $U \subset X$ *open* if for any $y \in U$ there exists $\epsilon > 0$ such that $B_\epsilon(y) \subset U$.

Problem 1.1. Proof that in a metric space (X, d), any ball $B_r(x)$ is an open set.

Problem 1.2. Let $\mathcal{F}$ be a family of open sets in a metric space X. Prove that the union

$$\bigcup_{A \in \mathcal{F}} A = \{x \in X : x \in A \text{ for at least one } A \in \mathcal{F}\}$$

is an open set.

Problem 1.3. Prove that the entire metric space itself is open.

Problem 1.4. Let (X, d) be a metric space, and let $x, y \in X$ be two distinct points. Prove that there exists two open sets U_x and U_y with $x \in U_x$ and $y \in U_y$, and such that $U_x \cap U_y = \emptyset$.

Problem 1.5. Let (X, d) be a metric space. Recall that a set $A \subset X$ is *closed* if for any sequence of points $\{x_n\}_{n \in \mathbb{N}} \subset X$ with $x_n \to y \in X$, we have that $y \in A$. Prove that if $A_1, A_2 \subset X$ are two disjoint closed sets, then for any point $x \in A_1$, there is an open set U containing x with $U \cap A_2 = \emptyset$.

2. Problems - Set theory and logic

Problem 2.1. Let A, B and C be sets. Prove the *distributive laws* that

$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$$

and

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C).$$

For a set A and a set B we define

$$A \setminus B = \{x \in A : x \notin B\}.$$

Problem 2.2. Prove that for two sets A and B we have that

$$(A \setminus B) \cap (B \setminus A) = \emptyset.$$

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Problem 2.3. Prove that for sets A, B and C we have that

$$A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C).$$
$$A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C).$$

(These are called deMorgan's laws.)

Recall the axioms of set theory that we have covered so far: extensionality, specification, pairing, power set, union, intersection, and the axiom of infinity, the last one postulating the existence of an infinite set

$$\mathbb{N} = \{0, 1, 2, 3, ...\}$$

Recall also that for $n \geq 2$ we have the notation

$$S_n = \{1, 2, ..., n-1\}.$$

Problem 2.4. Suppose that for a given $n \geq 2$ there is to each integer $j \in S_n$ assigned a set which we denote by $\alpha(j)$. Explain why you, based on the axioms considered so far, are allowed to construct the set

$$\{\alpha(1), ..., \alpha(n)\}.$$

Problem 2.5. Suppose that for each integer $j \in \mathbb{N}$ is assigned a set which we denote by $\alpha(j)$. Are you, based on the axioms considered so far, allowed to construct a set

$$\{\alpha(1), \alpha(2), \alpha(3), ...\}?$$

References

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