SUGGESTED SOLUTIONS - §31 – §35

§31

Problem 1 Let $x, y \in X$ with $x \neq y$. Since X is regular (and so also Hausdorff) there exist disjoint open sets U_x and U_y. Further there exist open sets V_x and V_y with $\bar{V}_x \subset U_x$ and $\bar{V}_y \subset U_y$ (Lemma 31.1).

Problem 2 Almost identical to the previous one.

Problem 5 We show that the complement is open. Suppose that $f(x) \neq g(x)$ for some $x \in X$. Since Y is Hausdorff there are disjoint open sets $U_{f(x)}$ and $U_{g(x)}$ in Y. Then $f^{-1}(U_{f(x)})$ and $g^{-1}(U_{g(x)})$ are open subsets in X and so

$$W_x = f^{-1}(U_{f(x)}) \cap g^{-1}(U_{g(x)}) \neq \emptyset$$

is an open set containing x, and we have that $f(y) \neq g(y)$ for all $y \in W_x$.

§32

Problem 1
Let X be normal and $Y \subset X$ closed. Suppose that $A, B \subset Y$ are closed and disjoint. Since Y is closed we have that A and B are also closed in X. So there are disjoint open sets U_A and U_B in X; so $U_A|_Y$ and $U_B|_Y$ are open sets in Y that separates A and B.

Problem 8

(a) Note that we have shown that such intervals are connected. Let $x_0 \in U$. Assume first that there is a point $y \in C$ with $y > x_0$. Then the set

$$\{x \in C : x > x_0\}$$

is a nonempty set which is bounded from below, and so it has a greatest lower bound c'. Then for any $d > c'$ there has to be a point $x \in C$ with $c' \leq x < d$, otherwise c' would not be a greatest lower bound (since there are points $c' < x < d$), and since C is closed, it follows that $c' \in C$. If there is a point $x \in C$ with $x < x_0$, a similar argument applies to show that there exists $c \in C$ such that $(c, c') \subset X \setminus C$; such an interval is a component of $X \setminus C$. If there is no such x it follows that $(-\infty, c')$ is a component. Finally, if there were no $x \in C$ with $x > x_0$ there would have to be some $x \in C$ with $x < x_0$, and a similar argument would show that (c, ∞) is a component for some $c \in C$.

(b) Assume that $x_0 \in \bar{C} \setminus C$. Then $x_0 \notin X \setminus (A \cup B)$, since any component contains at most one point from C, and this point is by assumption not x_0. Without loss of generality we may then assume $x_0 \in A$, and we note that $x_0 \notin \text{int}(A)$. Now for any interval (a, b) containing x_0 there exists some $c_W \in (a, b)$, and $c_W \in (c, c')$ which is a component of $X \setminus (A \cup B)$. We then have that (i) $c < c' < x_0$ or (ii) $x_0 < c < c'$. Without loss of generality we may assume that we always are in the case (i); for if there exists an interval

(a,b) such that there is no interval of case (i), for any subinterval we would be in case (ii). We now claim that there cannot exist $e < x_0$ such that c is always less than e, for if so (e, x_0) would be contained in a component which could only contain one single c_W. So for any $e < x_0$ there exist points x from both A and B with $e < x < x_0$, and it would follow that $x_0 \in \bar{B}$; a contradiction.

(c) We first show that C is nonempty. Let $a \in A$. Without loss of generality we assume that there exists a point $b \in B$ with $b > a$. Then the set

$$\{x \in B : x > a\}$$

in nonempty, it has a greater lower bound b, which is contained in B since B is closed. Applying a similar argument we may assume that a is the smallest upper bound for all $x \in A$ with $x < b$, and so $W = (a,b)$ is a component which contains a x_W. Finally, if an interval (c_{W_1}, c_{W_2}) contains points from both A and B, a similar argument would produce an interval $W \subset (c_{W_1}, c_{W_2})$ containing some c_W; a contradiction.

§33

Problem 2

(a) Let X be a normal countable space with at least two points x_1 and x_2. By Urysohn's Lemma there exists a continuous function $f : X \to [0,1]$ such that $f(x_0) = 0$ and $f(x_1) = 1$. If X is also connected it follows from the intermediate value theorem that for each $a \in [0,1]$ there exists $x \in X$ with $f(x) = a$, so since $[0,1]$ is uncountable we have that X is uncountable.

(b) Suppose that X is regular. We will show that if X is countable, then X is not connected. If X is a finite set, then each point is open, so we are done. So assume now that X is countably infinite, such that $X = \{x_n\}_{n \in \mathbb{Z}_+}$ with the x_n's pairwise disjoint Start by choosing two open sets U_1^1 and U_2^1 containing x_1 and x_2 respectively, and such that $\bar{U}_1^1 \cap \bar{U}_2^1 = \emptyset$. Assume now that for $n \in \mathbb{Z}_+$ we have constructed open sets $U_1^n, U_2^n, ..., U_m^n$ whose closures $\bar{U}_j^n$ are pairwise disjoint, and the for each $j \leq n$ we have that $x_j \in U_k^n$ for some k.

If there are no points x_m with $m > n$ such that x_m is not contained in U_k^n for some k we are done proving that X is not connected. Otherwise choose the smallest $m > n$ such that x_m is not contained in U_k^n for some k. There are now two possibilities. If $x_m \in \bar{U}_k^n$ choose an open set V containing x_m and such that $\bar{V} \cap \bar{U}_j^n$ for all $j \neq k$. Set $U_k^{n+1} = U_k^n \cup V$ and set $U_j^{n+1} = U_j^n$ for $j \neq k$. Now the $\bar{U}_i^{n+1}$'s are pairwise disjoint. The other possibility is that $x_m \notin \bar{U}_k^n$ for all k. In that case let U_{m+1}^{n+1} be an open set containing x_m such that $\bar{U}_{m+1}^{n+1} \cap \bar{U}_k^n = \emptyset$ for $k = 1, ..., m$, and relabel the U_k^n's by U_k^{n+1} for $k = 1, ..., m$.

Now for each $k \in \mathbb{Z}_+$ that appears in the construction (at least $k = 1, 2$ appears) we set $U_k = \cup_{n \geq N(k)} U_k^n$ (the $N(k)$ depends on when $U_k^{N(k)}$ first appears in the construction). Then the U_k's are pairwise disjoint open sets, and by the construction we have that $\{U_k\}$ covers X. Since $x_1 \in U_1$ and $x_2 \in U_2$ this shows that X is not connected.

Problem 4 Suppose first that such a function exists. Then we may set $U_n = \{x \in X : f(x) < \frac{1}{n}\}$, and we see that $A = \cap_n U_n$.

Suppose next that A is a G_δ set, and let $\{U_n\}_{n\in\mathbb{Z}_+}$ be collection of open sets such that $A = \cap_n U_n$. Since X is normal, for each $n \in \mathbb{Z}_+$ there exists by Urysohn's Lemma a continuous function $f_n : X \to [0, 2^{-n}]$ such that $f(x) = 0$ for all $x \in A$ and such that $f(x) = 2^{-n}$ for all $x \in X \setminus U_n$. Then

$$f(x) = \sum_{n=1}^{\infty} f_n(x)$$

converges uniformly to a desired continuous function.

§34

Problem 1 Consider Example 1 on page 197: the space $\mathbb{R}_K$ is Hausdorff but not regular. It also has a countable base for the topology since you can consider all intervals (p, q) with $p < q, p, q \in \mathbb{Q}$ and also all $(p, q) \setminus K$. This space is certainly not metrizable, since a metric space is even normal.

Problem 3 If X is a compact Hausdorff space, then it is automatically normal. So if it has a countable base it follows from the metrization theorem that it is metrizable.

Assume then that X is metrizable, i.e., we may equip X with a metric d which induces the topology on X. Then we have seen in class that if we for each $n \in \mathbb{Z}_+$ choose a collection $B_1^n, ..., B_{k_n}^n$ of balls of radius $1/n$ that covers X we get that $\mathcal{B} = \{B_k^n\}$ is a basis for the topology, and this family is countable.

§35

Problem 1 For two disjoint closed sets $A, B \subset$ define a continuous function $f : A \cup B \to [0, 1]$ by setting $f(x) = 0$ for all $x \in A$ and $f(x) = 1$ for all $x \in B$. Since B is closed we have that A is an open set in the subspace topology and vice versa, so $f^{-1}(a, b)$ is either A or B or $A \cup B$ or $\emptyset$, for any interval (a, b), and these are all open sets. By Tietze extension theorem we may extend f to a function on all of X.