SAMPLE SOLUTIONS FOR MAT3500/MAT4500 CONT. EXAM, JANUARY 2022

Problem 1

(a) … an open subset $U \subset X$ with $x \in U$.

(b) … for each open subset $V \subset Y$ the preimage $f^{-1}(V)$ is open in X.

(c) … there are no open, nonempty subsets $U, V \subset X$ with $U \cup V = X$ and $U \cap V = \emptyset$.

(d) A space is first countable if for each $x \in X$ there exists a countable collection $\mathcal{B} = \{B_n\}_{n=1}^{\infty}$ of neighborhoods of x such that for each neighborhood U of x there exists an n with $B_n \subset U$.

(e) … each singleton set in X is closed, and for any disjoint, closed subsets $A, B \subset X$ there exist disjoint, open subsets $U, V \subset X$ with $A \subset U$ and $B \subset V$.

(f) … a second-countable, Hausdorff space such that each point $x \in X$ has a neighborhood U that is homeomorphic to an open subset of $\mathbb{R}^m$.

Problem 2

(a) The Lebesgue number lemma: Let (X, d) be a compact metric space, and let $\mathcal{C}$ be an open cover of X. Then there exists a number $\delta > 0$ such that for each $A \subset X$ of diameter δ there exists a $U \in \mathcal{C}$ with $A \subset U$.

(b) The Tychonoff theorem: Any (finite or infinite) product of compact spaces is compact, in the product topology.

Problem 3

(a) Any compact subspace of a Hausdorff space is closed, so B is a closed subset of X. Hence $A \cap B$ is a closed subset of A, in the subspace topology from X. Any closed subset of a compact space is compact, so $A \cap B$ is compact in the subspace topology from A, which equals the subspace topology from X.

Problem 4

(a) By assumption, E is nonempty, so we can choose $e_0 \in E$ and let $b_0 = p(e_0)$. For any $b \in B$ there exists a path $f\colon [0,1] \to B$ from b_0 to B, since B is simply-connected and therefore path connected. By the path lifting property, there exists a path $g\colon [0,1] \to E$ from e_0 lifting f. Let $e = g(1)$. Then $p(e) = f(1) = b$. Hence p is surjective.

To prove that p is injective, suppose that $p(e_0) = b_0 = p(e_1)$. The lifting correspondence

$$\phi\colon \pi_1(B, b_0) \longrightarrow p^{-1}(b_0)$$

is surjective, since E is path connected. Its image $p^{-1}(b_0)$ has only a single element, since $\pi_1(B, b_0)$ is trivial group. Hence $e_0 = e_1$.

(b) Yes. Each covering map $p\colon E \to B$ is continuous and open. We know by (4a) that p is a bijection. Hence p is a homeomorphism.

[Proof that p is an open map, i.e., that for each open subset $W \subset E$ the image $p(W) \subset B$ is open. Let $b \in p(W)$. Since p is a covering map, there is an evenly covered neighborhood U of b. Hence $p^{-1}(U) = \coprod_\alpha V_\alpha$, with each $V_\alpha \subset E$ an open subset mapping homeomorphically to U. Since p is bijective, $p^{-1}(b)$ consists of a single element e, so the coproduct consists of a single summand V. Thus $V \subset E$ is open, and $p|V\colon V \to U$ is a homeomorphism. Since W is open in E the intersection $V \cap W$ is open in V, so $p(V \cap W) = U \cap p(W)$ is open in U. Since U is open in B, $U \cap p(W)$ is also open in B. Hence this is a neighborhood of b in B, contained in $p(W)$. Since b was arbitrarily chosen in $p(W)$, it follows that $p(W)$ is open in B.]

Problem 5

(a) Yes. For each good R the space X/R is Hausdorff, so Y is a product of Hausdorff spaces, and is therefore Hausdorff. Any subspace of Y is thus a subspace of a Hausdorff space, and is therefore Hausdorff.

(b) The function $f\colon X \to Y$ is continuous, because Y has the product topology and each component $\pi_R \circ f = q_R$ is continuous. Its corestriction $g\colon X \to Z$ is then continuous, because Z has the subspace topology from Y. It is surjective, because $g(X) = f(X) = Z$. Its image is a subspace of Y, hence is Hausdorff by (5a).

(c) Let $Q \subset X \times X$ be the equivalence relation defined by $(x, y) \in Q$ if and only if $g(x) = g(y)$. Then $g = k \circ \pi_Q$ where $k\colon X/Q \to Z$ is the continuous bijection with $k([x]) = g(x)$. Since Z is Hausdorff, so is X/Q, hence Q is good. The restricted projection $\ell = \pi_Q|Z\colon Z \to X/Q$ is also a continuous bijection. Hence k and ℓ are mutually inverse homeomorphisms, which implies that g is a quotient map.