SAMPLE SOLUTIONS FOR MAT3500/MAT4500 FINAL EXAM, AUTUMN 2021

Problem 1

(a) A basis for a topology on a set X is a collection $\mathcal{B}$ of subsets of X such that (1) for each $x \in X$ there is a $B \in \mathcal{B}$ with $x \in B$, and (2) for any two $B_1, B_2 \in \mathcal{B}$ and any $x \in B_1 \cap B_2$ there is a $B_3 \in \mathcal{B}$ with $x \in B_3 \subset B_1 \cap B_2$.

(b) The closure of a subset A of a topological space X is the intersection of all the closed subset that contain A.

(c) A homeomorphism $f\colon X \to Y$ is a continuous bijection such that the inverse function $f^{-1}\colon Y \to X$ is also continuous.

(d) A space X is locally connected if for each $x \in X$ and each neighborhood U of x there is a connected neighborhood V of x with $V \subset U$.

(e) A space X is compact if for each open cover $\mathcal{C} = \{U_\alpha\}_{\alpha \in J}$ of X there exists a finite subcover $\mathcal{F} = \{U_{\alpha_1}, \dots, U_{\alpha_n}\}$ of $\mathcal{C}$.

(f) A space X is regular if each singleton set $\{x\}$ is closed, and if for each $x \in X$ and closed subset $B \subset X$ with $x \notin B$ there are disjoint open subsets $U, V \subset X$ with $x \in U$ and $B \subset V$.

Problem 2

(a) The Urysohn Lemma: Let X be a normal space, and let $A, B \subset X$ be disjoint closed subset. Then there exists a continuous function $f\colon X \to [0,1]$ such that $f(x) = 0$ for all $x \in A$ and $f(x) = 1$ for all $x \in B$.

(b) The Urysohn Metrization Theorem: Let X be a regular and second-countable space. Then X is metrizable.

Problem 3

(a) Let $h\colon A \to (\mathbb{R} - \{0\}) \times \mathbb{R} \times \mathbb{R}$ be given by $h(a,b,c,d) = (a,b,c)$, with inverse $h^{-1}(a,b,c) = (a,b,c,(bc+1)/a)$. Each of these is continuous, because each component function is continuous (since $a \neq 0$). Hence h is a homeomorphism from A to the open subset $(\mathbb{R} - \{0\}) \times \mathbb{R} \times \mathbb{R}$ of $\mathbb{R}^3$.

Let $k\colon B \to \mathbb{R} \times (\mathbb{R} - \{0\}) \times \mathbb{R}$ be given by $k(a,b,c,d) = (a,b,d)$, with inverse $k^{-1}(a,b,d) = (a,b,(ad-1)/b,d)$. Each of these is continuous, because each component function is continuous (since $b \neq 0$). Hence k is a homeomorphism from B to the open subset $\mathbb{R} \times (\mathbb{R} - \{0\}) \times \mathbb{R}$ of $\mathbb{R}^3$.

(b) Yes, X is a 3-manifold. It is second-countable and Hausdorff because it is a subspace of the second-countable and Hausdorff space $\mathbb{R}^4$, and it is locally homeomorphic to $\mathbb{R}^3$ by (a). This uses that $A \cup B = X$, which holds because there are no points $(a,b,c,d) \in X$ with both $a = 0$ and $b = 0$.

(c) No, X is not compact. The function $f\colon X \to \mathbb{R}$ given by $f(a,b,c,d) = b$ is continuous. If X were compact, then its image $f(X)$ would be compact. However, f is surjective, since $(1,b,0,1) \in X$ for all $b \in \mathbb{R}$, so $f(X) = \mathbb{R}$ in the metric topology, which is not compact.

Problem 4

(a) If $f(w) \neq 0$ for all $w \in D^2$, then

$$r(z) = \frac{f(z)}{|f(z)|}$$

defines a continuous function $r\colon D^2 \to S^1$ with $r(z) = z/1 = z$ for each $z \in S^1$. Then $r \circ i = id$, where $i\colon S^1 \to D^2$ is the inclusion. Hence the composition

$$\pi_1(S^1,1) \xrightarrow{i_*} \pi_1(D^2,1) \xrightarrow{r_*} \pi_1(S^1,1)$$

is the identity. However, this is impossible, since $\pi_1(S^1,1)$ is nontrivial (an infinite cyclic group) and $\pi_1(D^2,1)$ is the trivial group. Hence $f(w) = 0$ for some $w \in D^2$.

Problem 5

(a) If $x \in U$ then $p(x) \in p(U)$ so $x \in p^{-1}(p(U))$. Hence $U \subset p^{-1}(p(U))$, with no assumption on U.

If U, V is a separation of E, then U is open and closed in E. For each $b \in B$ the intersection $F_b \cap U$ is then open and closed in $F_b = p^{-1}(b)$. Since F_b is connected, we must have $F_b \cap U = \emptyset$ or $F_b \cap U = F_b$.

Suppose that $x \in p^{-1}(p(U))$. Then $p(x) \in p(U)$, so that $p(x) = p(y)$ for some $y \in U$. Let $b = p(y)$. Then $y \in U$ and $y \in F_b$, so $F_b \cap U \neq \emptyset$. The only alternative is $F_b \cap U = F_b$. From $x \in F_b = F_b \cap U$, we deduce $x \in U$, so that $p^{-1}(p(U)) \subset U$. Hence $p^{-1}(p(U)) = U$, as required.

(b) Suppose that U, V is a separation of E. Then U is open in E, and $p^{-1}(p(U)) = U$ by (a). Since p is a quotient map, it follows that $p(U)$ is open in B. The same argument, with V in place of U, shows that $p(V)$ is open in B.

The union $p(U) \cup p(V) = p(U \cup V) = p(E)$ is equal to B, because p is surjective.

The intersection $p(U) \cap p(V)$ is empty, because otherwise there is a point $p(z) \in p(U)$ with $z \in V$. Then $z \in p^{-1}(p(U)) = U$, so $z \in U \cap V$. But U and V are disjoint.

Hence $p(U), p(V)$ is a separation of B. Since B is connected, this cannot happen. Hence no separation of E exists, so E is connected.